Microwave Filter Design

Chp3. Basic Concept and Theories of Filters

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Transfer Functions

General Definitions

- Transfer function: a mathematical description of network response characteristics.
- The transfer function of a 2-port filter network is usually defined as:

$$|S_{21}(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 F_n^2(\Omega)}$$

where ε is the *ripple constant*, *Fn*(Ω) is the *characteristic function*, and Ω is the *frequency variable* of a lowpass prototype filter that has a cutoff frequency $\Omega = \Omega_c$ for $\Omega_c = 1$ (rad/s).

- For linear, time invariant networks, the transfer function may be defined as a rational function:

$$S_{21}(p) = \frac{N(p)}{D(p)}$$

where *N(p)* and *D(p)* are polynomials in a complex frequency variable $p = \sigma + j\Omega$. For a lossless passive network, the neper frequency (damping coefficient)σ =0 and $p = j\Omega$.

Transfer Functions: Definitions

Insertion Loss

$$L_A(\Omega) = 10 \log \frac{1}{|S_{21}(j\Omega)|^2} \text{ dB}$$

Return Loss

$$|S_{11}|^2 + |S_{21}|^2 = 1 \text{ for a lossless, passive two-port network,}$$

$$L_R(\Omega) = 10 \log[1 - |S_{21}(j\Omega)|^2] \text{ dB}$$

Phase Response

$$\phi_{21} = \text{Arg } S_{21}(j\Omega)$$

Group Delay Response

$$\tau_d(\Omega) = \frac{d\phi_{21}(\Omega)}{-d\Omega} \text{ seconds}$$

Transfer Functions:
Poles and Zeros on the Complex Plane

Complex Plane (p-plane) : (σ, Ω)

The values of *p* at which the function becomes zero are the zeros of the function, and the values of *p* at which the function becomes infinite are the singularities (usually the poles) of the function.

Therefore, the zeros of *S*21(*p*) are the roots of the numerator *N* (*p*) and the poles of *S*21(*p*) are the roots of denominator *D* (*p*).

These poles will be the natural frequencies of the filter.

For the filter to be stable, these natural frequencies must lie in the left half of the *p*-plane, or on the imaginary axis.

D(*p*) is a Hurwitz polynomial, i.e., its roots (or zeros) are in the inside of the left half-plane, or on the *j*-axis,

the roots (or zeros) of *N*(*p*) may occur anywhere on the entire complex plane. The zeros of *N* (*p*) are called finite-frequency transmission zeros of the filter.

Butterworth (Maximally Flat) Response

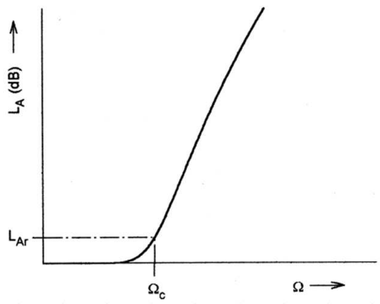
The amplitude-squared transfer function for Butterworth filters that have an insertion loss $L_{Ar} = 3.01$ dB at the cutoff frequency $\Omega_c = 1$ is given by

$$|S_{21}(j\Omega)|^2 = \frac{1}{1+\Omega^{2n}}$$

where n is the degree or the order of filter, which corresponds to the number of reactive elements required in the lowpass prototype filter.

Maximally flat because its amplitude-squared transfer function has the maximum number of $(2n-1)$ zero derivatives at $\Omega = 0$.

Therefore, the maximally flat approximation to the ideal lowpass filter in the passband is best at $\Omega = 0$, but deteriorates as Ω approaches the cutoff frequency Ω_c. Fig-

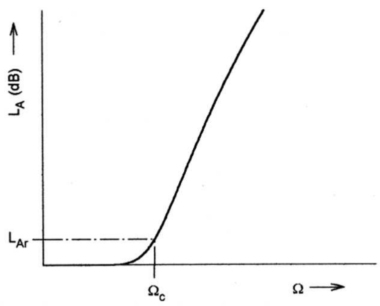


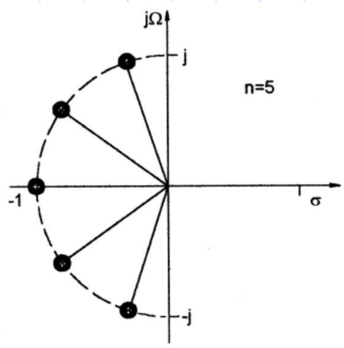
Butterworth (Maximally Flat) Response

A rational transfer function constructed from (3.7) is

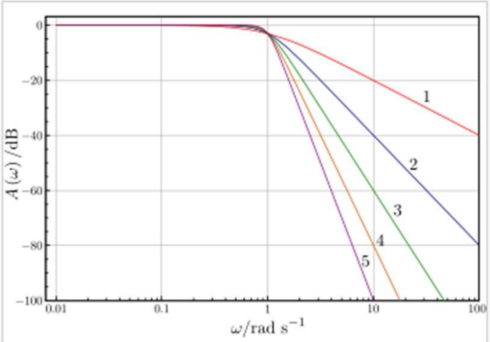
$$S_{21}(p) = \frac{1}{\prod_{i=1}^{n}(p-p_i)} \qquad p_i = j\exp\left[\frac{(2i-1)\pi}{2n}\right]$$

There is no finite-frequency transmission zero [all the zeros of $S_{21}(p)$ are at infinity], and the poles p_i lie on the unit circle in the left half-plane at equal angular spacings, since $|p_i| = 1$ and Arg $p_i = (2i-1)\pi/2n$. This is illustrated in Figure 3.2.





Order dependence



Plot of the gain of Butterworth low-pass filters of orders 1 through 5, with cutoff frequency $\omega_0 = 1$. Note that the slope is 20n dB/decade where n is the filter order.

Chebyshev Response

The Chebyshev response that exhibits the equal-ripple passband and maximally flat stopband is depicted in Figure 3.3. The amplitude-squared transfer function that describes this type of response is

$$|S_{21}(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_n^2(\Omega)} \tag{3.9}$$

where the ripple constant ε is related to a given passband ripple L_{Ar} in dB by

$$\varepsilon = \sqrt{10^{\frac{L_{Ar}}{10}} - 1} \tag{3.10}$$

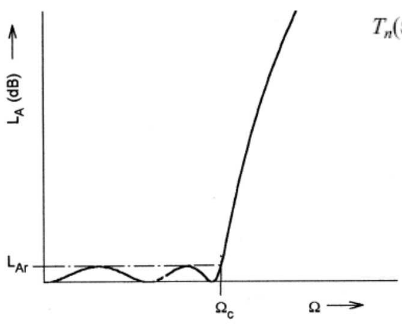


FIGURE 3.3 Chebyshev lowpass response.

$T_n(\Omega)$ is a Chebyshev function of the first kind of order n, which is defined

$$T_n(\Omega) = \begin{cases} \cos(n \cos^{-1} \Omega) & |\Omega| \le 1 \\ \cosh(n \cosh^{-1} \Omega) & |\Omega| \ge 1 \end{cases}$$

Chebyshev Polynomial

Chebyshev Differential Equations

$$(1-x^2)\,y'' - x\,y' + n^2\,y = 0$$

$$(1-x^2)\,y'' - 3x\,y' + n(n+2)\,y = 0$$

The solutions are the Chebyshev polynomials of the first and second kind, respectively

Chebyshev Polynomial of the first kind

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

The Chebyshev polynomials of the first kind can be defined by the trigonometric identity

$$T_n(x) = \cos(n \arccos x) = \cosh(n \operatorname{arccosh} x)$$

→ $$T_n(\cos(\vartheta)) = \cos(n\vartheta)$$

Chebyshev Polynomial of the second kind

$$U_0(x) = 1$$
$$U_1(x) = 2x$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

The Chebyshev polynomials of the second kind can be defined by the trigonometric identity

$$U_n(\cos(\vartheta)) = \frac{\sin((n+1)\vartheta)}{\sin\vartheta}$$

Chebyshev Response

Rhodes has derived a general formula of the rational transfer function

$$S_{21}(p) = \frac{\prod_{i=1}^{n}[\eta^2 + \sin^2(i\pi/n)]^{1/2}}{\prod_{i=1}^{n}(p+p_i)}$$

$$p_i = j\cos\left[\sin^{-1} j\eta + \frac{(2i-1)\pi}{2n}\right]$$

$$\eta = \sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)$$

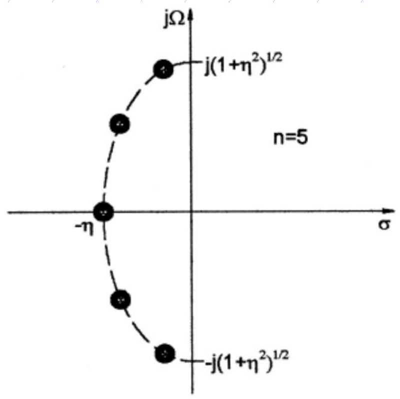


FIGURE 3.4 Pole distribution for Chebyshev response.

Similar to the maximally flat case, all the transmission zeros of $S21(p)$ are located at infinity. Therefore, the Butterworth and Chebyshev filters dealt with so far are sometimes referred to as all-pole filters.

The pole locations for the Chebyshev lie on an ellipse in the left half-plane.

Poles of Chebyshev Filter

Using the complex frequency *s*, the poles satisfy

$$1+\varepsilon^2 T_n^2(-js)=0$$

Defining $-js = \cos(\theta)$ and using the trigonometric definition of the Chebyshev polynomials yields

$$1+\varepsilon^2 T_n^2(\cos(\theta))=1+\varepsilon^2\cos^2(n\theta)=0.$$

Solving for θ

$$\theta=\frac{1}{n}\arccos\left(\frac{\pm j}{\varepsilon}\right)+\frac{m\pi}{n}$$

The poles of the Chebyshev gain function are then

$$\begin{aligned} s_{pm} &= j\cos(\theta) \\ &= j\cos\left(\frac{1}{n}\arccos\left(\frac{\pm j}{\varepsilon}\right)+\frac{m\pi}{n}\right) \end{aligned}$$

Using the properties of the trigonometric and hyperbolic functions, this may be written in explicitly complex form:

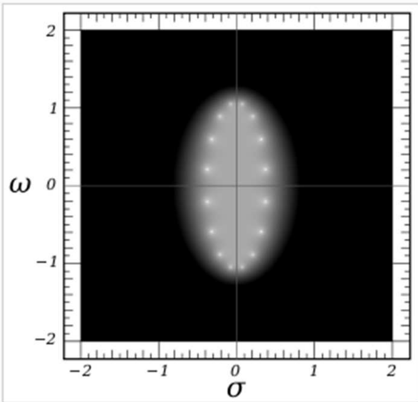
$$\begin{aligned} s_{pm}^{\pm} = &\pm\sinh\left(\frac{1}{n}\operatorname{arsinh}\left(\frac{1}{\varepsilon}\right)\right)\sin(\theta_m) \\ &+j\cosh\left(\frac{1}{n}\operatorname{arsinh}\left(\frac{1}{\varepsilon}\right)\right)\cos(\theta_m) \end{aligned}$$

where $m = 1, 2, \ldots, n$ and

$$\theta_m=\frac{\pi}{2}\,\frac{2m-1}{n}$$

It demonstrates that the poles lie on an **ellipse** in *s*-space centered at $s = 0$ with a real semi-axis of length $\sinh(\operatorname{arsinh}(1/\varepsilon)/n)$ and an imaginary semi-axis of length of $\cosh(\operatorname{arsinh}(1/\varepsilon)/n)$

Poles of Chebyshev Filter



Log of the absolute value of the gain of an 8th order Chebyshev type I filter in complex frequency space ($s = \sigma + j\omega$) with $\varepsilon = 0.1$ and $\omega_0 = 1$. The white spots are poles and are arranged on an ellipse with a semi-axis of 0.3836... in σ and 1.071... in ω. The transfer function poles are those poles in the left half plane. Black corresponds to a gain of 0.05 or less, white corresponds to a gain of 20 or more.

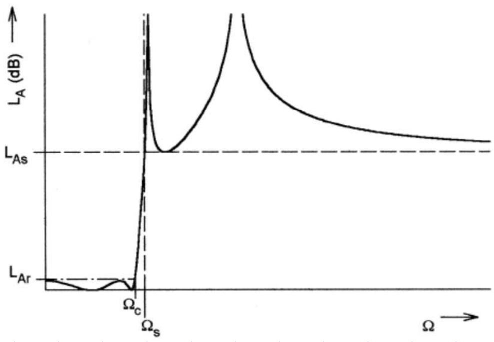
http://en.wikipedia.org/wiki/Chebyshev_filter

Elliptic Function Response

The response that is equal-ripple in both the passband and stopband is the elliptic function response, as illustrated in Figure 3.5. The transfer function for this type of response is

$$|S_{21}(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 F_n^2(\Omega)}$$

$$F_n(\Omega) = \begin{cases} M \dfrac{\prod_{i=1}^{n/2}(\Omega_i^2 - \Omega^2)}{\prod_{i=1}^{n/2}(\Omega_s^2/\Omega_i^2 - \Omega^2)} & \text{for } n \text{ even} \\ N \dfrac{\Omega \prod_{i=1}^{(n-1)/2}(\Omega_i^2 - \Omega^2)}{\prod_{i=1}^{(n-1)/2}(\Omega_s^2/\Omega_i^2 - \Omega^2)} & \text{for } n(\geq 3) \text{ odd} \end{cases}$$



where Ω_i $(0 < \Omega_i < 1)$ and $\Omega_s > 1$ represent some critical frequencies; M and N are constants to be defined

$F_n(\Omega)$ will oscillate between ± 1 for $|\Omega| \leq 1$, and $|F_n(\Omega = \pm 1)| = 1$.

Elliptic Function Response

Figure 3.6 plots the two typical oscillating curves for $n = 4$ and $n = 5$. Inspection of $F_n(\Omega)$ in (3.13b) shows that its zeros and poles are inversely proportional, the constant of proportionality being Ω_s. An important property of this is that if Ω_i can be found such that $F_n(\Omega)$ has equal ripples in the passband, it will automatically have equal ripples in the stopband. The parameter Ω_s is the frequency at which the equal-ripple stopband starts. For n even $F_n(\Omega_s) = M$ is required, which can be used to define the minimum in the stopband for a specified passband ripple constant ε.

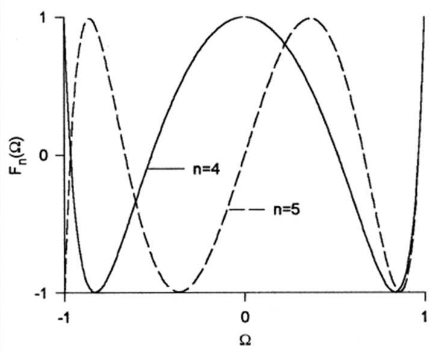


FIGURE 3.6 Plot of elliptic rational function.

Elliptic Filter

An elliptic filter (also known as a Cauer filter, named after Wilhelm Cauer) is a signal processing filter with equalized ripple (equiripple) behavior in both the passband and the stopband.

The amount of ripple in each band is independently adjustable, and no other filter of equal order can have a faster transition in gain between the passband and the stopband, for the given values of ripple (whether the ripple is equalized or not).

As the ripple in the stopband approaches zero, the filter becomes a type I Chebyshev filter.

As the ripple in the passband approaches zero, the filter becomes a type II Chebyshev filter

and finally, as both ripple values approach zero, the filter becomes a Butterworth filter.

Elliptic Filter

The gain of a lowpass elliptic filter as a function of angular frequency ω is given by:

$$G_n(\omega) = \frac{1}{\sqrt{1+\epsilon^2 R_n^2(\xi, \omega/\omega_0)}}$$

where R_n is the *n*th-order elliptic rational function and

ω_0 is the cutoff frequency

ε is the ripple factor

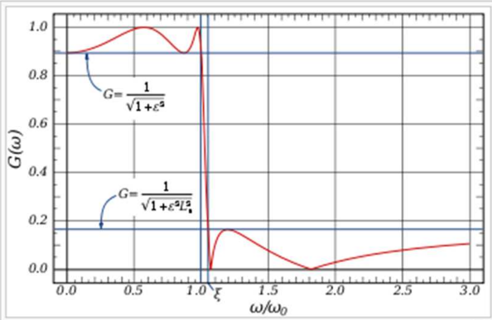
ξ is the selectivity factor

The passband of the gain therefore will vary between 1 and $1/\sqrt{1+\epsilon^2}$.

The gain of the stopband therefore will vary between 0 and $1/\sqrt{1+\epsilon^2 L_n^2}$.

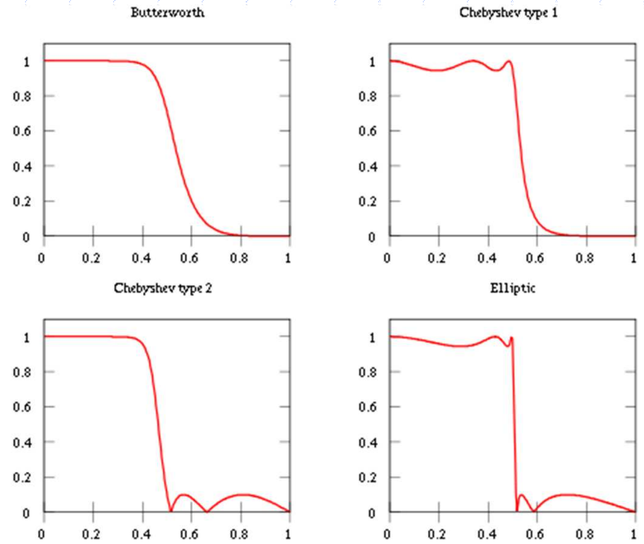
where $L_n = R_n(\xi, \xi)$

The value of the ripple factor specifies the passband ripple, while the combination of the ripple factor and the selectivity factor specify the stopband ripple.



The frequency response of a fourth-order elliptic low-pass filter with ε=0.5 and ξ=1.05. Also shown are the minimum gain in the passband and the maximum gain in the stopband, and the transition region between normalized frequency 1 and ξ

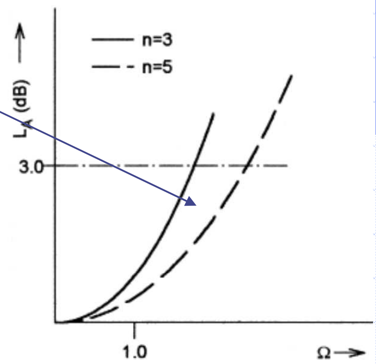
Elliptic Filter



Elliptic filters are sharper than all the others, but they show ripples on the whole bandwidth

Gaussian (Maximally Flat Group-Delay) Response

1. Poor selectivity
2. BW depends on order n ?



The Gaussian response is approximated by a rational transfer function

$$S_{21}(p) = \frac{a_0}{\sum_{k=0}^{n} a_k p^k}$$

where $p = \sigma + j\Omega$ is the normalized complex frequency variable, and the coefficients

$$a_k = \frac{(2n-k)!}{2^{n-k}k!(n-k)!} \qquad (3.15)$$

This transfer function posses a group delay that has maximum possible number of zero derivatives with respect to Ω at $\Omega = 0$, which is why it is said to have maximally flat group delay around $\Omega = 0$ and is in a sense complementary to the Butterworth response, which has a maximally flat amplitude.

Gaussian (Maximally Flat Group-Delay) Response

With increasing filter order n, the selectivity improves little and the insertion loss in decibels approaches the Gaussian form

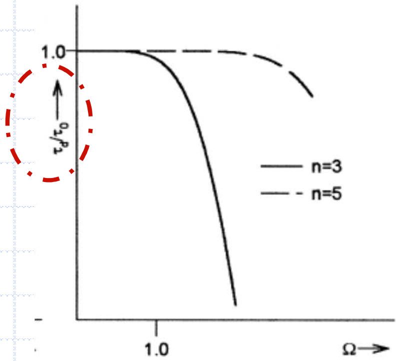
$$L_A(\Omega) = 10 \log e^{\frac{\Omega^2}{(2n-1)}} \text{ dB}$$

Use of this equation gives the 3 dB bandwidth as

$$\Omega_{3\text{ dB}} \approx \sqrt{(2n-1)\ln 2}$$

Unlike the Butterworth response, the 3 dB bandwidth of a Gaussian filter is a function of the filter order; the higher the filter order, the wider the 3 dB bandwidth.

Advantage: A quite flat group delay in the passband



All-pass Response

The transfer function of an all-pass network is defined by

$$S_{21}(p) = \frac{D(-p)}{D(p)}$$

where $p = \sigma + j\Omega$ is the complex frequency variable and $D(p)$ is a strict Hurwitz polynomial. At real frequencies ($p = j\Omega$), $|S_{21}(j\Omega)|^2 = S_{21}(p)S_{21}(-p) = 1$ so that the amplitude response is unity at all frequencies, which is why it is called the all-pass network.

However, there will be phase shift and group delay produced by the allpass network.

$$S_{21}(j\Omega) = e^{j\phi_{21}(\Omega)} \longrightarrow \phi_{21}(\Omega) = -j \ln S_{21}(j\Omega)$$

$$\tau_d(\Omega) = -\frac{d\phi_{21}(\Omega)}{d\Omega} = j\frac{d(\ln S_{21}(j\Omega))}{d\Omega}$$

$$\longrightarrow = j\left(\frac{1}{D(-p)}\frac{dD(-p)}{dp} - \frac{1}{D(p)}\frac{dD(p)}{dp}\right)\frac{dp}{d\Omega}\Bigg|_{p=j\Omega}$$

In mathematics, a Hurwitz polynomial, named after Adolf Hurwitz, is a polynomial whose coefficients are positive real numbers and whose zeros are located in the left half-plane of the complex plane, that is, the real part of every zero is negative.

All-pass Response

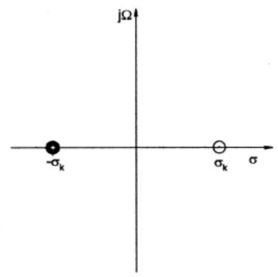
An expression for a strict Hurwitz polynomial $D(p)$ is

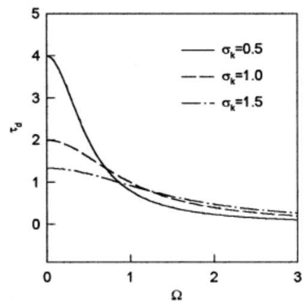
$$D(p) = \left(\prod_{k=1}^{n}[p-(-\sigma_k)]\right)\left(\prod_{k=1}^{m}[p-(-\sigma_i+j\Omega_i)]\cdot[p-(-\sigma_i-j\Omega_i)]\right)$$

If all poles and zeros of an all pass network are located along the -axis, such a network is said to consist of C-type sections and therefore referred to as C-type all-pass network.

If the poles and zeros of the transfer function are all complex with quadrantal symmetry about the origin of the complex plane, the resultant network is referred to as D-type all-pass network consisting of D-type sections only.

C-type (single section)





$$S_{21}(p) = \frac{-p+\sigma_k}{p+\sigma_k}$$

$$\tau_d(\Omega) = \frac{2\sigma_k}{\sigma_k^2+\Omega^2}$$

All-pass Response

D-type (single section)

$$S_{21}(p) = \frac{[-p-(-\sigma_i+j\Omega_i)]\cdot[-p-(-\sigma_i-j\Omega_i)]}{[p-(-\sigma_i+j\Omega_i)]\cdot[p-(-\sigma_i-j\Omega_i)]}$$

$$\tau_d(\Omega) = \frac{4\sigma_i[(\sigma_i^2+\Omega_i^2)+\Omega^2]}{[(\sigma_i^2+\Omega_i^2)-\Omega^2]^2+(2\sigma_i\Omega)^2}$$

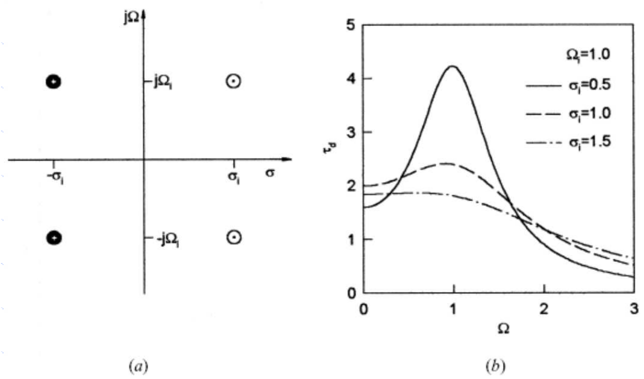


FIGURE 3.9 Characteristics of single-section, D-type, all-pass network: (a) pole-zero diagram, (b) group delay response.

Lowpass Prototype Filters

Filter syntheses for realizing the transfer functions usually result in the so-called lowpass prototype filters

lowpass prototype filter is in general defined as the lowpass filter whose element values are normalized to make the source resistance or conductance equal to one, denoted by $g_0 = 1$, and the cutoff angular frequency to be unity, denoted by $\Omega_c = 1$(rad/s).

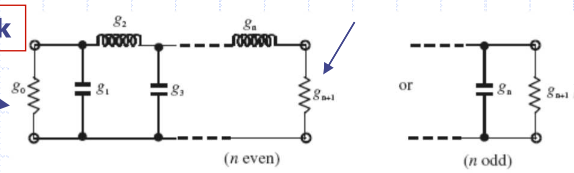
n-pole lowpass prototype for realizing an all-pole filter response, including Butterworth, Chebyshev, and Gaussian responses.

Ladder Network

source resistance

load conductance

load resistance

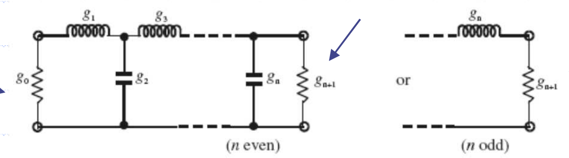


Dual

source conductance

load resistance

load conductance



These g-values are supposed to be the inductance in henries, capacitance in farads, resistance in ohms, and conductance in mhos.

Butterworth Lowpass Prototype Filters

$$g_0 = 1.0$$

$$g_i = 2\sin\left(\frac{(2i-1)\pi}{2n}\right) \quad \text{for } i = 1 \text{ to } n$$

$$g_{n+1} = 1.0$$

To determine the degree of a Butterworth lowpass prototype, a specification that is usually the minimum stopband attenuation L_{As} dB at $\Omega = \Omega_s$ for $\Omega_s > 1$ is given.

$$n \geq \frac{\log(10^{0.1L_{AS}} - 1)}{2\log\Omega_s}$$

For example, if $L_{As} = 40$ dB and $\Omega_s = 2$, $n \geq 6.644$, i.e., a 7-pole ($n = 7$) Butterworth prototype should be chosen.

TABLE 3.1 Element values for Butterworth lowpass prototype filters ($g_0 = 1.0$, $\Omega_c = 1$, $L_{Ar} = 3.01$ dB at Ω_c)

n	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
1	2.0000	1.0								
2	1.4142	1.4142	1.0							
3	1.0000	2.0000	1.0000	1.0						
4	0.7654	1.8478	1.8478	0.7654	1.0					
5	0.6180	1.6180	2.0000	1.6180	0.6180	1.0				
6	0.5176	1.4142	1.9318	1.9318	1.4142	0.5176	1.0			
7	0.4450	1.2470	1.8019	2.0000	1.8019	1.2470	0.4450	1.0		
8	0.3902	1.1111	1.6629	1.9616	1.9616	1.6629	1.1111	0.3902	1.0	
9	0.3473	1.0000	1.5321	1.8794	2.0000	1.8794	1.5321	1.0000	0.3473	1.0

Chebyshev Lowpass Prototype Filters

For Chebyshev lowpass prototype filters with a passband ripple L_{Ar} dB and the cutoff frequency $\Omega_c = 1$.

$$g_0 = 1.0$$

$$g_1 = \frac{2}{\gamma}\sin\left(\frac{\pi}{2n}\right)$$

$$g_i = \frac{1}{g_{i-1}} \frac{4\sin\left[\frac{(2i-1)\pi}{2n}\right]\cdot\sin\left[\frac{(2i-3)\pi}{2n}\right]}{\gamma^2 + \sin^2\left[\frac{(i-1)\pi}{n}\right]} \quad \text{for } i = 2, 3, \cdots n$$

$$g_{n+1} = \begin{cases} 1.0 & \text{for } n \text{ odd} \\ \coth^2\left(\frac{\beta}{4}\right) & \text{for } n \text{ even} \end{cases}$$

$$\beta = \ln\left[\coth\left(\frac{L_{Ar}}{17.37}\right)\right]$$

$$\gamma = \sinh\left(\frac{\beta}{2n}\right)$$

For passband ripple $L_{Ar} = 0.1$ dB

n	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
1	0.3052	1.0								
2	0.8431	0.6220	1.3554							
3	1.0316	1.1474	1.0316	1.0						
4	1.1088	1.3062	1.7704	0.8181	1.3554					
5	1.1468	1.3712	1.9750	1.3712	1.1468	1.0				
6	1.1681	1.4040	2.0562	1.5171	1.9029	0.8618	1.3554			
7	1.1812	1.4228	2.0967	1.5734	2.0967	1.4228	1.1812	1.0		
8	1.1898	1.4346	2.1199	1.6010	2.1700	1.5641	1.9445	0.8778	1.3554	
9	1.1957	1.4426	2.1346	1.6167	2.2054	1.6167	2.1346	1.4426	1.1957	1.0

Chebyshev Lowpass Prototype Filters

Using the same example as given above for the Butterworth prototype, i.e., $L_{As} \geq 40$ dB at $\Omega_s = 2$, but a passband ripple $L_{Ar} = 0.1$ dB for the Chebyshev response, we have $n \geq 5.45$, i.e., $n = 6$ for the Chebyshev prototype to meet this specification.

For the required passband ripple L_{Ar} dB, the minimum stopband attenuation L_{As} dB at $\Omega = \Omega_s$, the degree of a Chebyshev lowpass prototype, which will meet this specification, can be found by

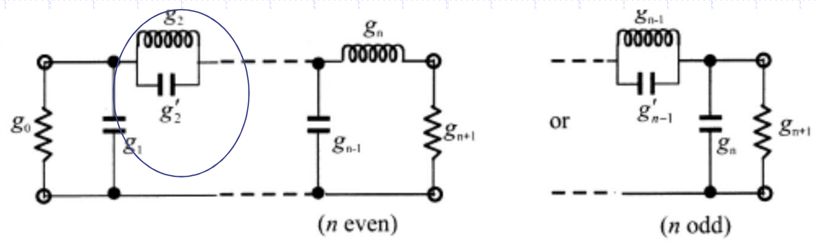
$$n \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1L_{As}} - 1}{10^{0.1L_{Ar}} - 1}}}{\cosh^{-1}\Omega_s}$$

Sometimes, the minimum return loss L_R or the maximum voltage standing wave ratio *VSWR* in the passband is specified instead of the passband ripple L_{Ar}. If the re-

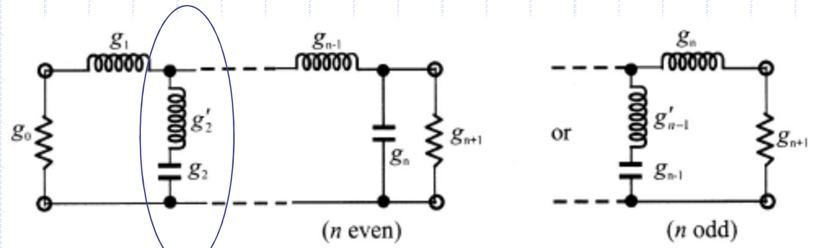
$$L_{Ar} = -10\log(1 - 10^{0.1L_R}) \text{ dB}$$

Elliptic Lowpass Prototype Filters

The series branches of parallel-resonant circuits are introduced for realizing the finite-frequency transmission zeros



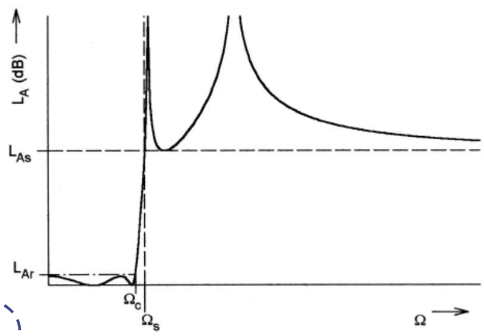
The shunt branches of series-resonant circuits are used for implementing the finite-frequency transmission zeros



Prof. T. L. Wu

Elliptic Lowpass Prototype Filters

Unlike the Butterworth and Chebyshev lowpass prototype filters, there is no simple formula available for determining element values of the elliptic function lowpass prototype filters.



instance, considering the same example as used above for the Butterworth and Chebyshev prototype, i.e., $L_{As} \geq 40$ dB at $\Omega_s = 2$ and the passband ripple $L_{Ar} = 0.1$ dB, we can determine immediately $n = 5$ by inspecting the design data, i.e., Ω_s and L_{As} listed in Table 3.3. This also shows that the elliptic function design is superior to both the Butterworth and Chebyshev designs for this type of specification.

Gaussian Lowpass Prototype Filters

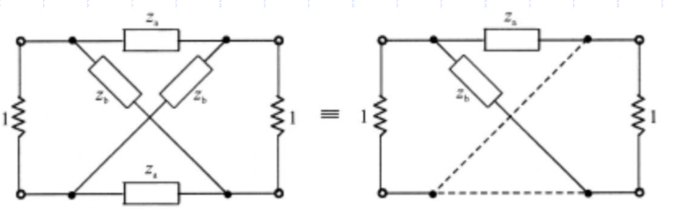
two useful design parameters. The first one is the value of Ω, denoted by $\Omega_{1\%}$, for which the group delay has fallen off by 1% from its value at $\Omega = 0$. Along with this parameter is the insertion loss at $\Omega_{1\%}$, denoted by $L_{\Omega 1\%}$ in dB. Not

TABLE 3.4 Element values for Gaussian lowpass prototype filters ($g_0 = g_{n+1} = 1.0$, $\Omega_s = 1$)

n	$\Omega_{1\%}$	$L_{\Omega 1\%}$ dB	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
2	0.5627	0.4794	1.5774	0.4226								
3	1.2052	1.3365	1.2550	0.5528	0.1922							
4	1.9314	2.4746	1.0598	0.5116	0.3181	0.1104						
5	2.7090	3.8156	0.9303	0.4577	0.3312	0.2090	0.0718					
6	3.5245	5.3197	0.8377	0.4116	0.3158	0.2364	0.1480	0.0505				
7	4.3575	6.9168	0.7677	0.3744	0.2944	0.2378	0.1778	0.1104	0.0375			
8	5.2175	8.6391	0.7125	0.3446	0.2735	0.2297	0.1867	0.1387	0.0855	0.0289		
9	6.0685	10.3490	0.6678	0.3203	0.2547	0.2184	0.1859	0.1506	0.1111	0.0682	0.0230	
10	6.9495	12.188	0.6305	0.3002	0.2384	0.2066	0.1808	0.1539	0.1240	0.0911	0.0557	0.0187

It is noteworthy that the higher order (n >= 5) Gaussian filters extend the flat group delay property into the frequency range where the insertion loss has exceeded 3 dB.

All-Pass, Lowpass Prototype Filters



By inspection

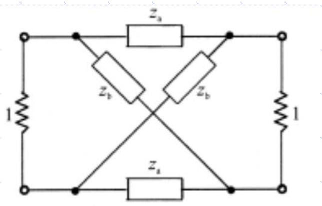
$$z_{11} = z_{22} = \frac{z_b + z_a}{2}$$

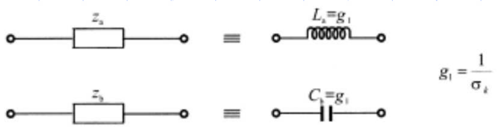
$$z_{12} = z_{21} = \frac{z_b - z_a}{2}$$

→ S-parameters can then be derived

All-Pass, Lowpass Prototype Filters

C-type





If the elements are assigned as

$$z_a = j\Omega L_a = j\Omega g_1$$

$$z_b = \frac{1}{j\Omega C_a} = \frac{1}{j\Omega g_1}$$

$$g_1 = \frac{1}{\sigma_k}$$

It can derived that

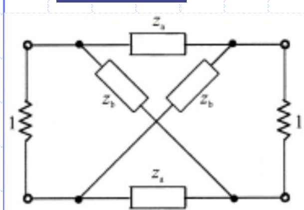
$$S_{21}(j\Omega) \equiv \frac{z_b - z_a}{z_{11}z_{22} + z_{11} + z_{22} - z_{12}z_{21} + 1} = \frac{j\Omega - 1/g_i}{j\Omega + 1/g_i} = \frac{p - \sigma_k}{p + \sigma_k}$$

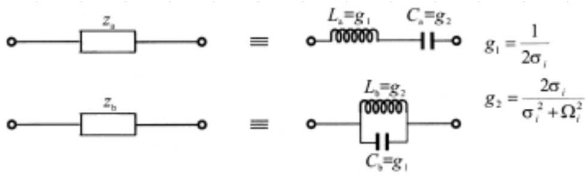
where $\sigma_k > 0$ is the design parameter that will control the group delay characteristics, as shown in Figure 3.8.

p. 14 in slides

All-Pass, Lowpass Prototype Filters

D-type





Similar to C-type, the elements are assigned as

$$z_a = j\Omega L_a + \frac{1}{j\Omega C_a}, \qquad \frac{1}{z_b} = j\Omega C_b + \frac{1}{j\Omega L_b}$$

$$L_a = C_b = g_1 = \frac{1}{2\sigma_i}, \qquad C_a = L_b = g_2 = \frac{2\sigma_i}{\sigma_i^2 + \Omega_i^2}$$

where $\sigma_i > 0$ and $\Omega_i > 0$ are the two design parameters that will shape the group delay response, as illustrated in Figure 3.9. Since a D-type section is the second-order all-pass network, there are actually two lowpass prototype elements, namely g_1 and g_2, which will represent both the inductance of an inductor and the capacitance of a capacitor, depending on the locations of these reactive elements, as indicated in Figure 3.12(c).

FREQUENCY AND ELEMENT TRANSFORMATIONS

- Frequency and impedance scaling
- Lowpass Transformation
- Highpass Transformation
- Bandpass Transformation
- Bandpass Transformation

Impedance and frequency scaling

- Normalized source resistance/conductance $\rightarrow$ $g_0 = 1$
- Cutoff frequency $\rightarrow$ $\Omega_c = 1$
  - impedance scaling : define an impedance scaling factor γ_0

$$\gamma_0 = \begin{cases} Z_0/g_0 & \text{for } g_0 \text{ being the resistance} \\ g_0/Y_0 & \text{for } g_0 \text{ being the conductance} \end{cases}$$

$$\text{Scaled value : impedance} \rightarrow \text{impedance} \times \gamma_0$$

$$L \rightarrow \gamma_0 L \qquad C \rightarrow C/\gamma_0$$

$$R \rightarrow \gamma_0 R \qquad G \rightarrow G/\gamma_0$$

The impedance scaling will remove the $g_0 = 1$ normalization and adjust the filter to work for any value of the source impedance denoted by $Z0$. This scaling will has no effect on the response shape.

Resistive element transformation for the generic term g for the lowpass prototype elements in the element transformation

$$R = \gamma_0 g \quad \text{for } g \text{ representing the resistance}$$

$$G = \frac{g}{\gamma_0} \quad \text{for } g \text{ representing the conductance}$$

Impedance and frequency scaling

♥ Frequency scaling :

$$\Omega = \left(\frac{\Omega_c}{\omega_c}\right)\omega \Rightarrow \left|S'_{21}(j\Omega)\right|^2 = \left|S_{21}\left(j\frac{\Omega_c}{\omega_c}\omega\right)\right|^2$$

scaled values :

$$j\Omega L = j\left(\frac{\Omega_c}{\omega_c}\right)\omega L \Rightarrow L' = \left(\frac{\Omega_c}{\omega_c}\right)L \qquad \frac{1}{j\Omega C} = \frac{1}{j\left(\frac{\Omega_c}{\omega_c}\right)\omega C} \Rightarrow C' = \left(\frac{\Omega_c}{\omega_c}\right)C$$

♥ Impedance and Frequency scaling :

scaled values :

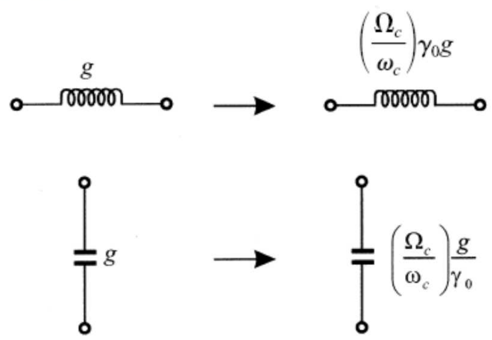
$$L = \left(\frac{\Omega_c}{\omega_c}\right)\gamma_0 g \quad \text{for } g \text{ representing the inductance}$$

$$C = \left(\frac{\Omega_c}{\omega_c}\right)\frac{g}{\gamma_0} \quad \text{for } g \text{ representing the capacitance}$$

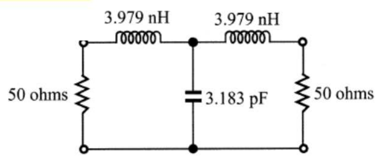
Lowpass Transformation

$$L = \left(\frac{\Omega_c}{\omega_c}\right)\gamma_0 g \quad \text{for } g \text{ representing the inductance}$$

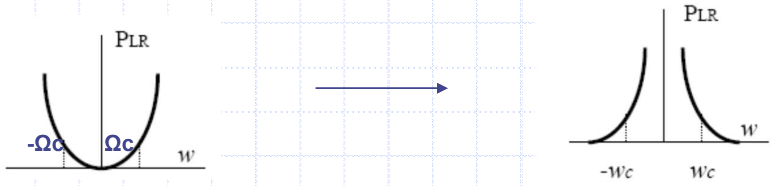
$$C = \left(\frac{\Omega_c}{\omega_c}\right)\frac{g}{\gamma_0} \quad \text{for } g \text{ representing the capacitance}$$



Example: To demonstrate the use of the element transformation, let us consider design of a practical lowpass filter with a cutoff frequency f_c = 2 GHz and a source impedance Z_0 = 50 ohms. A 3-pole Butterworth lowpass prototype with the structure of Figure 3.10(*b*) is chosen for this example, which gives $g_0 = g_4 = 1.0$ mhos, $g_1 = g_3 = 1.0$ H, and $g_2 = 2.0$ F for $\Omega_c = 1.0$ rad/s, from Table 3.1. The impedance scaling factor is $\gamma_0 = 50$, according to (3.34). The angular cutoff frequency $\omega_c = 2\pi \times 2 \times 10^9$ rad/s. Applying (3.38), we find $L_1 = L_3 = 3.979$ nH and $C_2 = 3.183$ pF. The resultant lowpass filter is illustrated in Figure 3.13(*b*).



Highpass Transformation



$$\because 0 \to \pm\infty, \Omega_c \to -\omega_c, -\Omega_c \to \omega_c \Rightarrow \Omega = -\frac{\omega_c \Omega_c}{\omega}$$

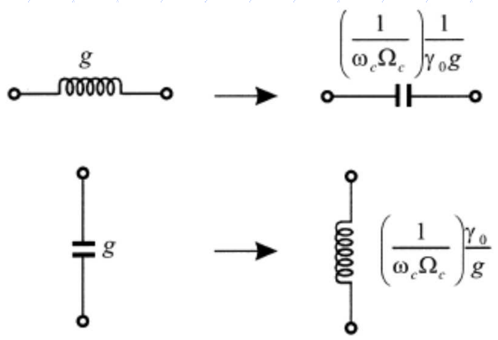
$$\frac{1}{j\Omega \dfrac{C}{\gamma_0}} = \frac{1}{j\left(-\dfrac{\omega_c \Omega_c}{\omega}\right)\dfrac{C}{\gamma_0}} = j\omega L' \Rightarrow L' = \left(\frac{1}{\omega_c \Omega_c}\right)\frac{\gamma_0}{C}$$

$$j\Omega \gamma_0 L = j\left(-\frac{\omega_c \Omega_c}{\omega}\right)\gamma_0 L = \frac{1}{j\omega C'} \Rightarrow C' = \left(\frac{1}{\omega_c \Omega_c}\right)\frac{1}{\gamma_0 L}$$

Highpass Transformation

An inductive/capacitive element in the lowpass prototype will be inversely transformed to a capacitive/inductive element in the highpass filter.

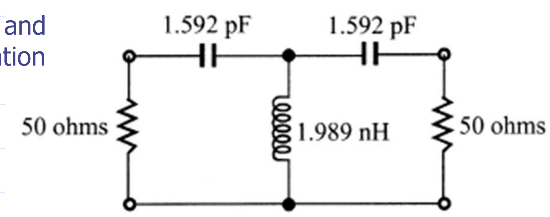
$$\Omega = -\frac{\omega_c \Omega_c}{\omega} \longrightarrow j\Omega g \to \frac{\omega_c \Omega_c g}{j\omega}$$



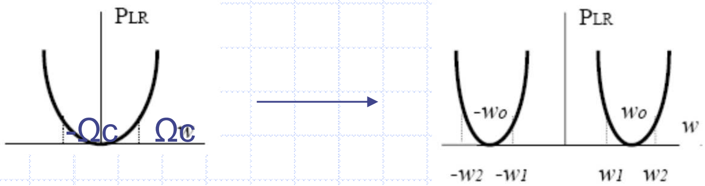
$$C = \left(\frac{1}{\omega_c \Omega_c}\right)\frac{1}{\gamma_0 g} \quad \text{for } g \text{ representing the inductance}$$

$$L = \left(\frac{1}{\omega_c \Omega_c}\right)\frac{\gamma_0}{g} \quad \text{for } g \text{ representing the capacitance}$$

Example:
A practical highpass filter with a cutoff frequency at 2GHz and 50-ohms terminals, which is obtained from the transformation of the 3 pole Butterworth lowpass prototype given above.



Bandpass Transformation



$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

$$FBW = \frac{\omega_2 - \omega_1}{\omega_0}$$

$$0 \to \omega_0,\ \Omega_c \to \omega_2,\ -\Omega_c \to \omega_1,\ \Omega = \frac{\Omega_c}{FBW}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

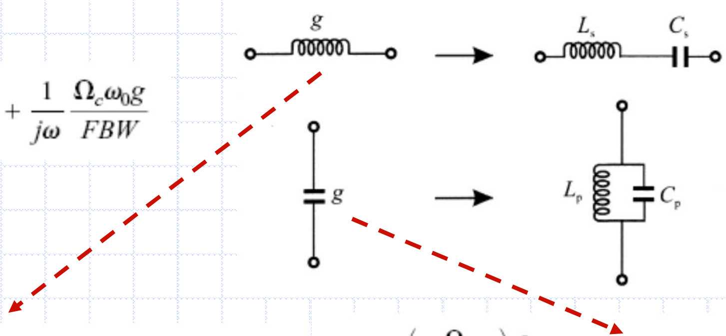
$$\frac{1}{j\Omega \frac{C}{\gamma_0}} = \frac{1}{j\frac{\Omega_c}{FBW}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\frac{C}{\gamma_0}} = \frac{1}{j\omega\left(\frac{\Omega_c}{FBW\,\omega_0}\right)\frac{C}{\gamma_0} + \frac{1}{j\omega\left(\frac{FBW}{\omega_0\Omega_c}\right)\frac{\gamma_0}{C}}} \Rightarrow C_p = \left(\frac{\Omega_c}{FBW\,\omega_0}\right)\frac{C}{\gamma_0}, L_p = \left(\frac{FBW}{\omega_0\Omega_c}\right)\frac{\gamma_0}{C}$$

$$j\Omega\gamma_0 L = j\frac{\Omega_c}{FBW}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\gamma_0 L = j\omega\left(\frac{\Omega_c}{FBW\,\omega_0}\right)\gamma_0 L + \frac{1}{j\omega\left(\frac{FBW}{\omega_0\Omega_c}\right)\frac{1}{\gamma_0 L}} \Rightarrow C_S = \left(\frac{FBW}{\omega_0\Omega_c}\right)\frac{1}{\gamma_0 L}, L_S = \left(\frac{\Omega_c}{FBW\,\omega_0}\right)\gamma_0 L$$

Bandpass Transformation

An inductive/capacitive element *g* in the lowpass prototype will transform to a series/parallel *LC* resonant circuit in the bandpass filter.

$$j\Omega g \to j\omega\frac{\Omega_c g}{FBW\omega_0} + \frac{1}{j\omega}\frac{\Omega_c \omega_0 g}{FBW}$$



$$L_s = \left(\frac{\Omega_c}{FBW\omega_0}\right)\gamma_0 g$$

$$C_s = 1/(\omega_0^2 L_s)$$

$$C_p = \left(\frac{\Omega_c}{FBW\omega_0}\right)\frac{g}{\gamma_0}$$

$$L_p = 1/(\omega_0^2 C_p)$$

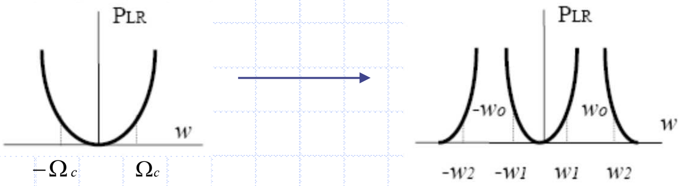
$$L_s = \left(\frac{\Omega_c}{FBW\omega_0}\right)\gamma_0 g \qquad C_s = \left(\frac{FBW}{\omega_0\Omega_c}\right)\frac{1}{\gamma_0 g}$$

for *g* representing the inductance

$$C_p = \left(\frac{\Omega_c}{FBW\omega_0}\right)\frac{g}{\gamma_0} \qquad L_p = \left(\frac{FBW}{\omega_0\Omega_c}\right)\frac{\gamma_0}{g}$$

for *g* representing the capacitance

Bandstop Transformation



$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

$$FBW = \frac{\omega_2 - \omega_1}{\omega_0}$$

$$0 \to \pm\infty,\ \ \Omega_c \to w_1,\ \ -\Omega_c \to w_2 \Rightarrow \Omega = \frac{\Omega_c FBW}{\left(\omega_0/\omega - \omega/\omega_0\right)}$$

$$j\Omega\gamma_0 L = j\frac{\Omega_c FBW}{\left(\omega_0/\omega - \omega/\omega_0\right)}\gamma_0 L = \frac{1}{\frac{\omega_0}{j\omega\Omega_c FBW\gamma_0 L} + j\omega\frac{1}{FBW\omega_0\Omega_c\gamma_0 L}} \Rightarrow C_p = \left(\frac{1}{FBW\omega_0\Omega_c}\right)\frac{1}{\gamma_0 L}, L_p = \left(\frac{\Omega_c FBW}{\omega_0}\right)\gamma_0 L$$

$$\frac{1}{j\Omega\frac{C}{\gamma_0}} = \frac{1}{j\frac{\Omega_c FBW}{\left(\omega_0/\omega - \omega/\omega_0\right)}\frac{C}{\gamma_0}} = \frac{1}{j\omega\left(\frac{\Omega_c FBW}{\omega_0}\right)\frac{C}{\gamma_0}} + j\omega\frac{1}{FBW\omega_0\Omega_c}\frac{\gamma_0}{C} \Rightarrow C_s = \left(\frac{\Omega_c FBW}{\omega_0}\right)\frac{C}{\gamma_0}, L_s = \frac{1}{FBW\omega_0\Omega_c}\frac{\gamma_0}{C}$$

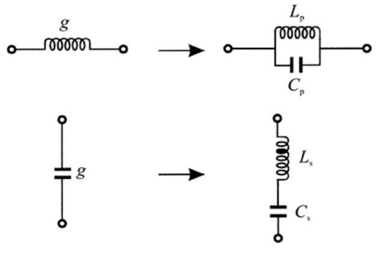
Bandstop Transformation

Opposite to the bandpass transformation, an inductive/capacitive element *g* in the lowpass prototype will transform to a parallel/series *LC* resonant circuit in the bandstop filter.

$$\Omega = \frac{\Omega_c FBW}{(\omega_0/\omega - \omega/\omega_0)}$$

$$\omega_0 = \sqrt{\omega_1\omega_2}$$

$$FBW = \frac{\omega_2 - \omega_1}{\omega_0}$$



$$L_p = \left(\frac{\Omega_c FBW}{\omega_0}\right)\gamma_0 g$$

$$C_p = 1/(\omega_0^2 L_p)$$

$$L_s = 1/(\omega_0^2 C_s)$$

$$C_s = \left(\frac{\Omega_c FBW}{\omega_0}\right)\frac{g}{\gamma_0}$$

(*a*)

$$C_p = \left(\frac{1}{FBW\omega_0\Omega_c}\right)\frac{1}{\gamma_0 g}$$

$$L_p = \left(\frac{\Omega_c FBW}{\omega_0}\right)\gamma_0 g$$

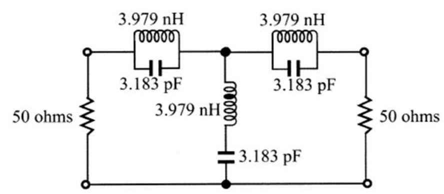
for *g* representing the inductance

$$L_s = \left(\frac{1}{FBW\omega_0\Omega_c}\right)\frac{\gamma_0}{g}$$

$$C_s = \left(\frac{\Omega_c FBW}{\omega_0}\right)\frac{g}{\gamma_0}$$

for *g* representing the capacitance

Ex:



IMMITTANCE INVERTERS

- Definition of Immittance, Impedance, and Admittance Inverters
- Filters with Immittance Inverters
- Practical Realization of Immittance Inverters

Definition of Immittance

- Immittance inverters are either impedance or admittance inverters.
- An idealized impedance inverter is a two-port network that has a unique property at all frequencies
- if it is terminated in an impedance $Z2$ on one port, the impedance $Z1$ seen looking in at the other port is

$$Z_1 = \frac{K^2}{Z_2}$$

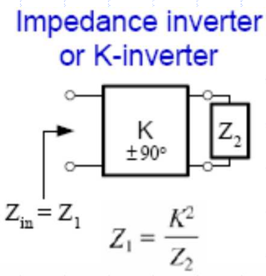
K: characteristic impedance of the inverter

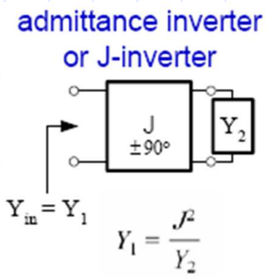
- As can be seen, if $Z2$ is inductive/conductive, $Z1$ will become conductive/inductive, and hence the inverter has a phase shift of ±90 degrees or an odd multiple thereof.

Impedance/admittance Inverter

- the inverter has a phase shift of ±90 degrees
- Impedance/admittance inverters are also known as *K*-inverters / J-inverters.
- The *ABCD* matrix of ideal impedance inverters may generally be expressed as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & \mp jK \\ \pm \dfrac{1}{jK} & 0 \end{bmatrix} \qquad \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & \pm \dfrac{1}{jJ} \\ \mp jJ & 0 \end{bmatrix}$$

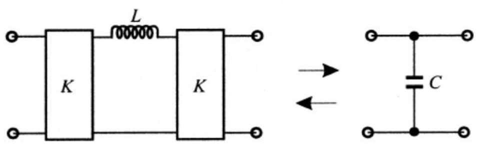




It is noted that the +/- 90 degree is decided by the phase of $\mathbf{S_{21}}$ of the 2-port network

Filters with Immittance Inverters

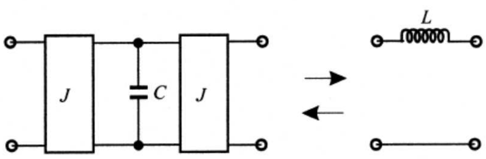
A series inductance with an inverter on each side looks like a shunt capacitance from its exterior terminals



Proved by ABCD matrix

$$\begin{bmatrix} 0 & jK \\ -\dfrac{1}{jK} & 0 \end{bmatrix}\begin{bmatrix} 1 & j\omega L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & jK \\ -\dfrac{1}{jK} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ j\omega C & 1 \end{bmatrix}$$

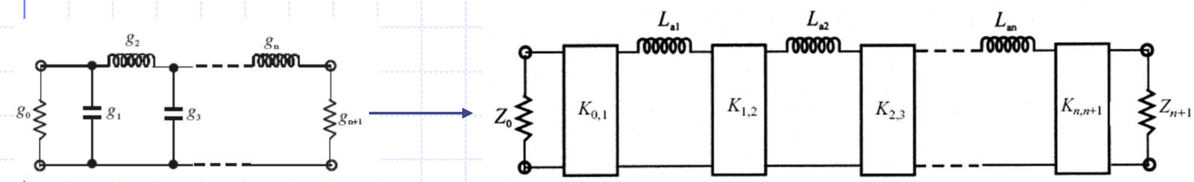
A shunt capacitance with an inverter on each side looks likes a series inductance from its external terminals



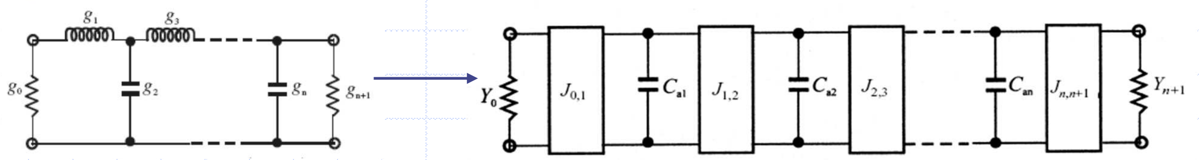
Inverters have the ability to shift impedance or admittance levels depending on the choice of *K* or *J* parameters.

Making use of these properties enables us to convert a filter circuit to an equivalent form that would be more convenient for implementation with microwave structures.

Lowpass Filters with Impedance Inverters



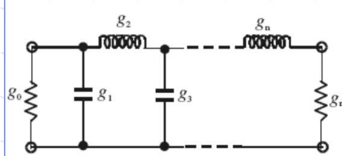
$$K_{0,1}=\sqrt{\frac{Z_0L_{a1}}{g_0g_1}}\ ,\quad K_{i,i+1}=\sqrt{\frac{L_{ai}L_{a(i+1)}}{g_ig_{i+1}}}\Bigg|_{i=1\text{ to }n-1}\ ,\quad K_{n,n+1}=\sqrt{\frac{L_{an}Z_{n+1}}{g_ng_{n+1}}}$$

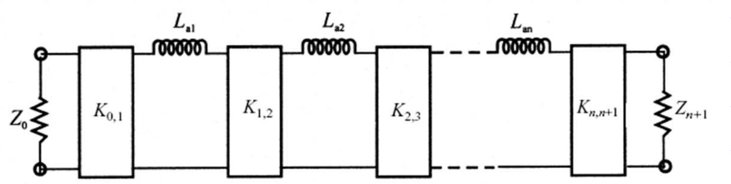


$$J_{0,1}=\sqrt{\frac{Y_0C_{a1}}{g_0g_1}}\ ,\quad J_{i,i+1}=\sqrt{\frac{C_{ai}C_{a(i+1)}}{g_ig_{i+1}}}\Bigg|_{i=1\text{ to }n-1}\ ,\quad J_{n,n+1}=\sqrt{\frac{C_{an}Y_{n+1}}{g_ng_{n+1}}}$$

The new element values, such as Z_0, Z_{n+1}, L_{ai}, Y_0, Y_{n+1}, and C_{ai} may be chosen arbitrarily and the filter response will be identical to that of the original prototype, provided that the immittance inverter parameters $K_{i,i+1}$ and $J_{i,i+1}$ are specified as indicated by the equations in Figure 3.18.

Lowpass Filters with Admittance Inverters





$$K_{0,1}=\sqrt{\frac{Z_0L_{a1}}{g_0g_1}}\ ,\quad K_{i,i+1}=\sqrt{\frac{L_{ai}L_{a(i+1)}}{g_ig_{i+1}}}\Bigg|_{i=1\text{ to }n-1}\ ,\quad K_{n,n+1}=\sqrt{\frac{L_{an}Z_{n+1}}{g_ng_{n+1}}}$$

How to prove?

Take n=2 as an example

By expanding the input immittances of the original prototype networks and the equivalent ones in continued fractions and by equating corresponding terms.

$$Z(s)=\cfrac{1}{\cfrac{1}{g_3+sg_2}+g_1s}$$

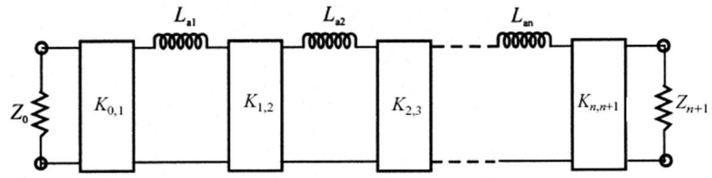
Compare their coefficients

$$\tilde{Z}(s)=\cfrac{K_{01}^2}{\cfrac{K_{12}^2}{sL_{a2}+\cfrac{K_{23}^2}{Z_3}}+sL_{a1}}$$

As an example

$$\frac{g_1}{(Z_0/g_0)}=\frac{L_{a1}}{K^2{}_{01}}\qquad \therefore K_{01}=\sqrt{\frac{L_{a1}Z_0}{g_1g_0}}$$

Bandpass Filters with Impedance Inverters



$$K_{0,1}=\sqrt{\frac{Z_0L_{a1}}{g_0g_1}}\ ,\quad K_{i,i+1}=\sqrt{\frac{L_{ai}L_{a(i+1)}}{g_ig_{i+1}}}\Bigg|_{i=1\text{ to }n-1}\ ,\quad K_{n,n+1}=\sqrt{\frac{L_{an}Z_{n+1}}{g_ng_{n+1}}}$$

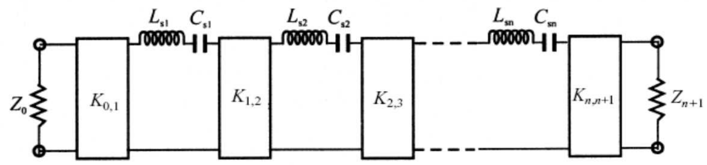
Since the source impedances are assumed the same in the both filters as indicated, no impedance scaling is required.

transforming the inductors of the lowpass filter to the series resonators of the bandpass filter, we obtain:

Replace L_{a1}

$$L_{si}=\left(\frac{\Omega_c}{FBW\omega_0}\right)L_{ai}$$

$$C_{si}=\frac{1}{\omega_0^2L_{si}}$$

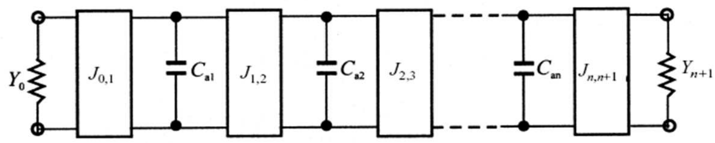


$$K_{0,1}=\sqrt{\frac{Z_0FBW\omega_0L_{s1}}{\Omega_cg_0g_1}}\ ,\quad K_{i,i+1}=\frac{FBW\omega_0}{\Omega_c}\sqrt{\frac{L_{si}L_{s(i+1)}}{g_ig_{i+1}}}\Bigg|_{i=1\text{ to }n-1}\ ,\quad K_{n,n+1}=\sqrt{\frac{FBW\omega_0L_{sn}Z_{n+1}}{\Omega_cg_ng_{n+1}}}$$

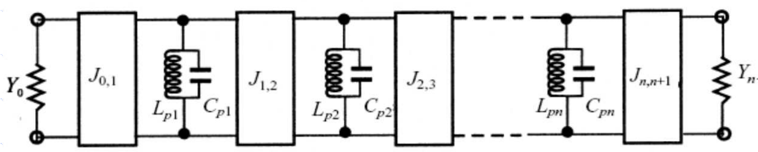
$$C_{si}=\frac{1}{\omega_0^2L_{si}}\Bigg|_{i=1\text{ to }n}$$

Bandpass Filters with admittance Inverters

Similarly



$$J_{0,1}=\sqrt{\frac{Y_0C_{a1}}{g_0g_1}}\ ,\quad J_{i,i+1}=\sqrt{\frac{C_{ai}C_{a(i+1)}}{g_ig_{i+1}}}\Bigg|_{i=1\text{ to }n-1}\ ,\quad J_{n,n+1}=\sqrt{\frac{C_{an}Y_{n+1}}{g_ng_{n+1}}}$$



$$J_{0,1}=\sqrt{\frac{Y_0FBW\omega_0C_{p1}}{\Omega_cg_0g_1}}\ ,\quad J_{i,i+1}=\frac{FBW\omega_0}{\Omega_c}\sqrt{\frac{C_{pi}C_{p(i+1)}}{g_ig_{i+1}}}\Bigg|_{i=1\text{ to }n-1}\ ,\quad J_{n,n+1}=\sqrt{\frac{FBW\omega_0C_{pn}Y_{n+1}}{\Omega_cg_ng_{n+1}}}$$

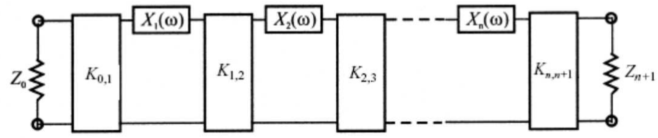
$$L_{pi}=\frac{1}{\omega_0^2C_{pi}}\Bigg|_{i=1\text{ to }n}$$

Generalized bandpass filters (including distributed elements) using immittance inverters.

Reactance slope parameter for resonators having zero reactance at center frequency

$$x = \frac{\omega_0}{2} \frac{dX(\omega)}{d\omega}\bigg|_{\omega=\omega_0}$$

a lumped LC series resonator is $\omega_0 L$.



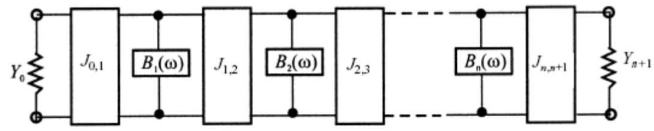
$$K_{0,1} = \sqrt{\frac{Z_0 FBW x_1}{\Omega_c g_0 g_1}}, \quad K_{i,i+1} = \frac{FBW}{\Omega_c}\sqrt{\frac{x_i x_{i+1}}{g_i g_{i+1}}}\bigg|_{i=1 \text{ to } n-1}, \quad K_{n,n+1} = \sqrt{\frac{FBW x_n Z_{n+1}}{\Omega_c g_n g_{n+1}}}$$

$$x_i = \frac{\omega_0}{2} \frac{dX_i(\omega)}{d\omega}\bigg|_{\omega=\omega_0}$$

Susceptance slope parameter for resonators having zero susceptance at center frequency

$$b = \frac{\omega_0}{2} \frac{dB(\omega)}{d\omega}\bigg|_{\omega=\omega_0}$$

a lumped LC parallel resonator is $\omega_0 C$.

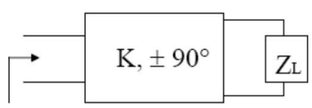


$$J_{0,1} = \sqrt{\frac{Y_0 FBW b_1}{\Omega_c g_0 g_1}}, \quad J_{i,i+1} = \frac{FBW}{\Omega_c}\sqrt{\frac{b_i b_{i+1}}{g_i g_{i+1}}}\bigg|_{i=1 \text{ to } n-1}, \quad J_{n,n+1} = \sqrt{\frac{FBW b_n Y_{n+1}}{\Omega_c g_n g_{n+1}}}$$

$$b_i = \frac{\omega_0}{2} \frac{dB_i(\omega)}{d\omega}\bigg|_{\omega=\omega_0}$$

Practical Realization of Immittance Inverters

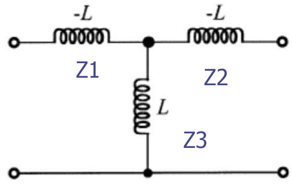
impedance inverter



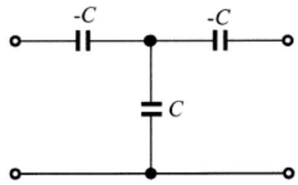
$$Z_{in} = \frac{K^2}{Z_L}$$

$$K, \pm\frac{\lambda}{4}$$

proof



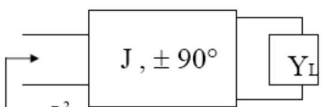
$K=\omega L$ +90⁰



$K = \frac{1}{\omega C}$ -90⁰

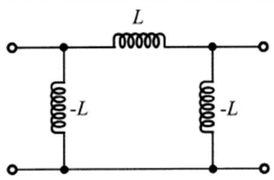
$$\begin{bmatrix} 1+\frac{Z_1}{Z_3} & Z_1+Z_2+\frac{Z_1 Z_2}{Z_3} \\ \frac{1}{Z_3} & 1+\frac{Z_2}{Z_3} \end{bmatrix} = \begin{bmatrix} 0 & -j\omega L \\ \frac{1}{j\omega L} & 0 \end{bmatrix}$$

admittance inverter

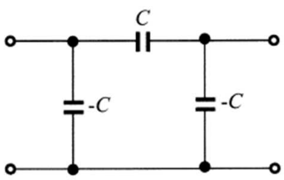


$$Y_{in} = \frac{J^2}{Y_L}$$

$$\frac{1}{J}, \frac{\lambda}{4}$$



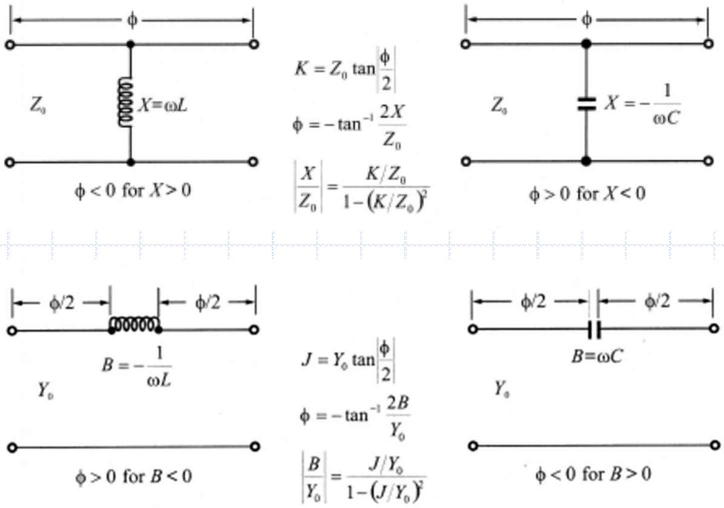
$J = \frac{1}{\omega L}$ -90⁰



$J=\omega C$ +90⁰

Immittance inverters comprised of lumped and transmission line elements

- A circuit mixed with lumped and transmission line elements



Richards' Transformation

Distributed transmission line elements are of importance for designing practical microwave filters.

A commonly used approach to the design of a practical distributed filter is to seek some approximate equivalence between lumped and distributed elements.

Such equivalence can be established by applying Richards's transformation

Richards showed that distributed networks, comprised of commensurate length (equal electrical length) transmission lines and lumped resistors, could be treated in analysis or synthesis as lumped element *LCR networks under the transformation*

The new complex plane where *t is defined is called the t-plane.*
Following is referred to as Richards' transformation

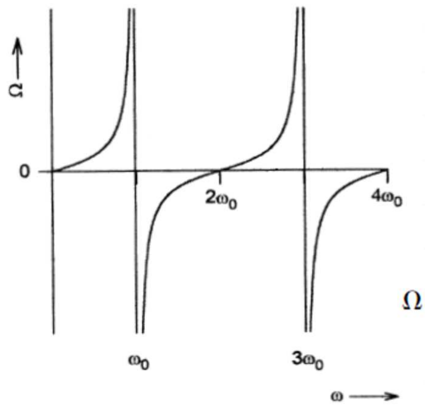
$$t = \tanh\frac{lp}{v_p}, \quad p = \sigma + j\omega$$

lossless passive networks

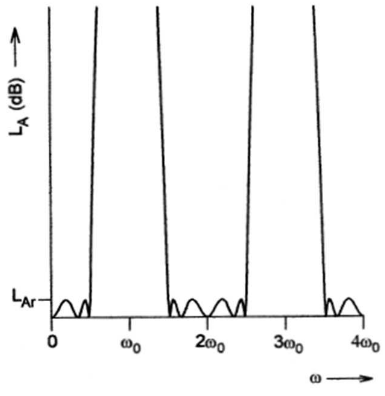
$$t = j\tan\theta, \quad p = j\omega$$

$$\theta = \frac{\omega}{v_p} l = \text{the electrical length}$$

Richards' Transformation



$$\Omega = \tan\left(\frac{\pi}{2}\frac{\omega}{\omega_0}\right)$$



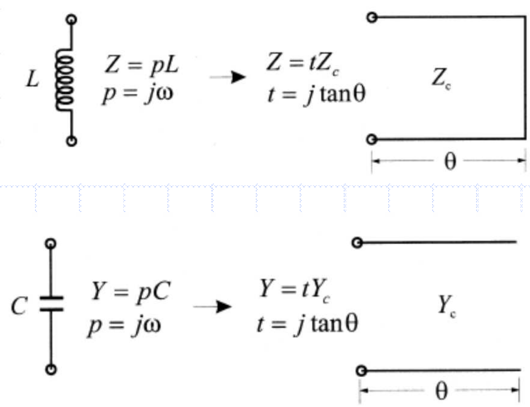
(*b*)

The periodic frequency response of the distributed filter network is demonstrated in Figure 3.23(*b*), *which is obtained by applying the Richards' transformation of (3.54) to the* Chebyshev lowpass prototype transfer function of (3.9), showing that the response repeats in frequency intervals of $2w_0$.

A lowpass response in the *p-plane may be mapped into either the lowpass* or the bandstop one in the *t-plane, depending on the design objective. Similarly,* it can be shown that a highpass response in the *p-plane may be transformed as either* the highpass or the bandpass one in the *t-plane.*

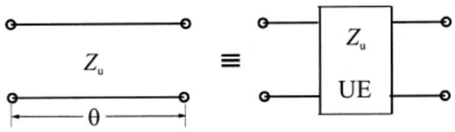
Richards' Transformation

Lumped and distributed element correspondence under Richards' transformation



Richards' Transformation

Another important distributed element, which has no lumped-element counterpart, is a two-port network consisting of a commensurate-length line.



ABCD matrix: $\begin{bmatrix} \cos\theta & jZ_u \sin\theta \\ j\sin\theta / Z_u & \cos\theta \end{bmatrix}$ $\quad t = j\tan\theta \quad$ ABCD matrix: $\frac{1}{\sqrt{1-t^2}}\begin{bmatrix} 1 & Z_u t \\ t/Z_u & 1 \end{bmatrix}$

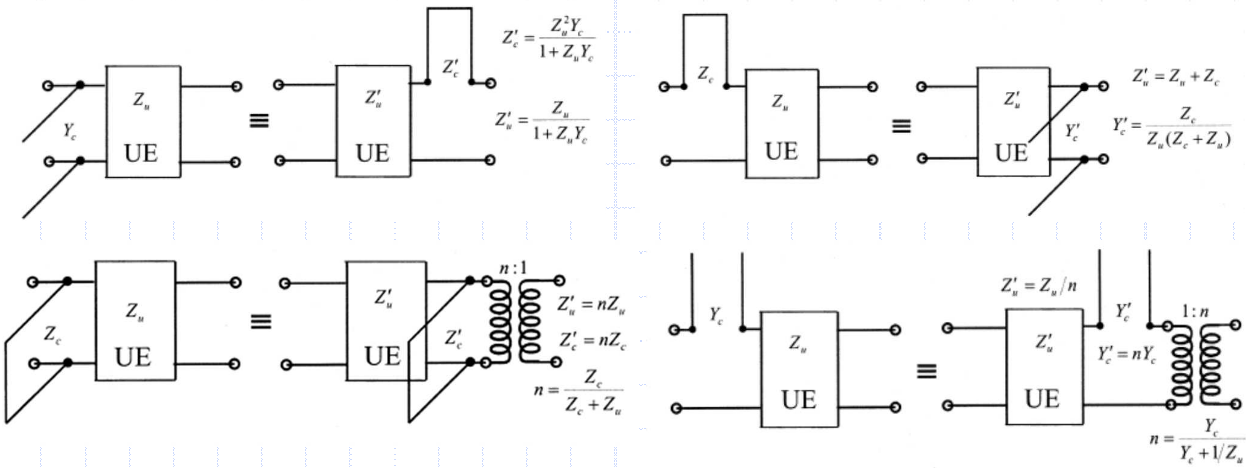
It is interesting to note that the **unit element** has a half-order transmission zero at $t = \pm 1$.

Unit elements are usually employed to separate the circuit elements in distributed filters, which are otherwise located at the same physical point.

Kuroda Identities

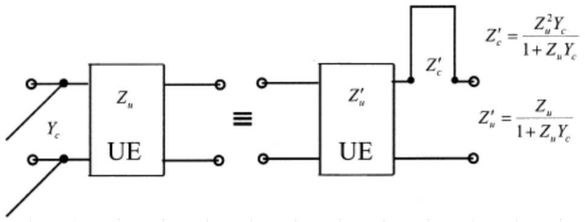
Such transformations not only provide designers with flexibility, but also are essential in many cases to obtain networks that are physically realizable with physical dimensions.

- Physically separate transmision line stubs
- Transform series stubs into shunt stubs, or vice versa
- Change impractical characteristic impedances into more realizable ones



Kuroda Identities

Proof:

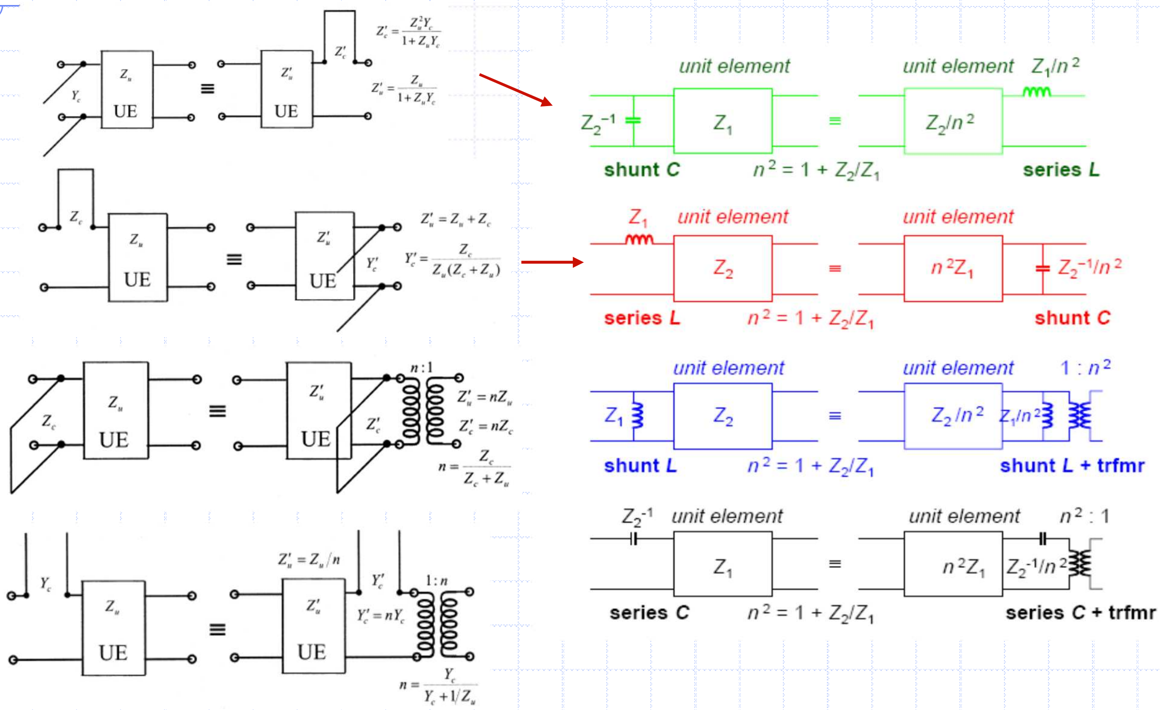


$$\frac{1}{\sqrt{1-t^2}}\begin{bmatrix}1 & 0\\ Y_c t & 1\end{bmatrix}\begin{bmatrix}1 & Z_u t\\ t/Z_u & 1\end{bmatrix} = \frac{1}{\sqrt{1-t^2}}\begin{bmatrix}1 & Z_u t\\ (Y_c+1/Z_u)t & Y_c Z_u t^2+1\end{bmatrix}$$

$\longleftrightarrow$

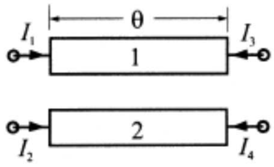
$$\frac{1}{\sqrt{1-t^2}}\begin{bmatrix}1 & Z_u' t\\ t/Z_u' & 1\end{bmatrix}\begin{bmatrix}1 & Z_c' t\\ 0 & 1\end{bmatrix} = \frac{1}{\sqrt{1-t^2}}\begin{bmatrix}1 & (Z_u'+Z_c')t\\ t/Z_u' & \frac{Z_c'}{Z_u'}t^2+1\end{bmatrix}$$

Kuroda identities

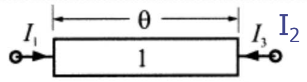


Coupled-Line Equivalent Circuits

Coupled line



Single line



Similar form

$$\begin{bmatrix}V_1\\V_2\\V_3\\V_4\end{bmatrix} = \frac{v_p}{t}\begin{bmatrix}[L] & \sqrt{1-t^2}[L]\\ \sqrt{1-t^2}[L] & [L]\end{bmatrix}\cdot\begin{bmatrix}I_1\\I_2\\I_3\\I_4\end{bmatrix}$$

$$\begin{bmatrix}V_1\\V_2\end{bmatrix} = \begin{bmatrix}-jZ_0\cot\theta & -jZ_0\csc\theta\\ -jZ_0\csc\theta & -jZ_0\cot\theta\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix}$$

$$= \frac{v_p}{t}\begin{bmatrix}L & L/\sqrt{1-t^2}\\ \sqrt{1-t^2}L & L\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix}$$

$$[L] = \begin{bmatrix}L_{11} & L_{12}\\ L_{12} & L_{22}\end{bmatrix}$$

Proof: refer to "Microwave Engineering"

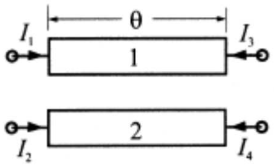
In this case L_{11} and L_{22} are the self-inductance per unit length of line 1 and line 2, respectively, and L_{12} is the mutual inductance per unit length. If the coupled lines are symmetrical, $L_{11} = L_{22}$. It may be remarked that $[L]$ and $[C]$ together satisfy

$$[L]\cdot[C] = [C]\cdot[L] = [U]/v_p^2 \qquad (3.58)$$

where $[U]$ denotes the identity matrix.

Coupled-Line Equivalent Circuits

Admittance matrix form

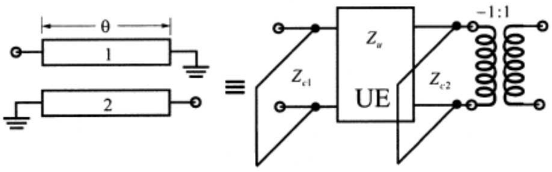


$$\begin{bmatrix}I_1\\I_2\\I_3\\I_4\end{bmatrix} = \frac{v_p}{t}\begin{bmatrix}[C] & -\sqrt{1-t^2}[C]\\ -\sqrt{1-t^2}[C] & [C]\end{bmatrix}\cdot\begin{bmatrix}V_1\\V_2\\V_3\\V_4\end{bmatrix}$$

$$[C] = \begin{bmatrix}C_{11} & -C_{12}\\ -C_{12} & C_{22}\end{bmatrix}$$

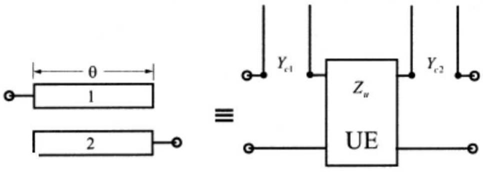
C_{11} and C_{22} are the self-capacitance per unit length of line 1 and line 2 respectively, and C_{12} is the mutual capacitance per unit length. Note that $C_{11} = C_{22}$ if the coupled lines are symmetrical.

Coupled-Line Equivalent Circuits

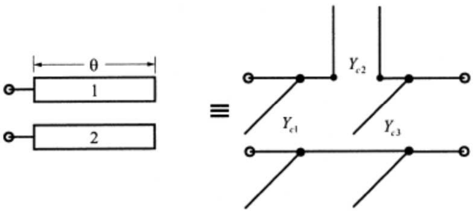


$$Z_{c1} = \frac{1}{v_p(C_{11} - C_{12})}$$
$$Z_{c2} = \frac{1}{v_p(C_{22} - C_{12})}$$
$$Z_u = 1/(v_p C_{12})$$

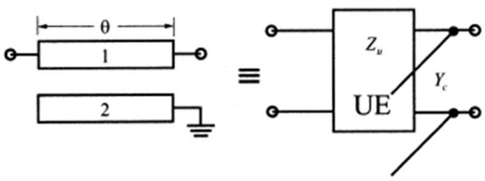
It can be proved by ABCD parameters



$$Y_{c1} = \frac{1}{v_p(L_{11} - L_{12})}$$
$$Y_{c2} = \frac{1}{v_p(L_{22} - L_{12})}$$
$$Z_u = v_p L_{12}$$



$$Y_{c1} = v_p(C_{11} - C_{12})$$
$$Y_{c2} = v_p C_{12}$$
$$Y_{c3} = v_p(C_{22} - C_{12})$$



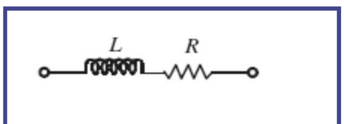
$$Z_u = v_p L_{11}$$
$$Y_c = \frac{v_p C_{12}^2}{C_{22}}$$

Dissipation and unloaded quality factor

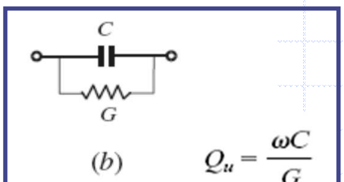
In reality, any practical microwave filter will have lossy elements with finite unloaded quality factors in association with power dissipation in these elements.

Such parasitic dissipation may frequently lead to substantial differences between the response of the filter actually realized and that of the ideal one designed with lossless elements.

Unloaded quality factor

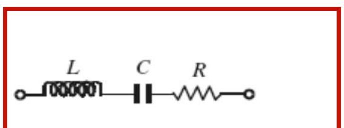


$$Q_u = \frac{\omega L}{R}$$

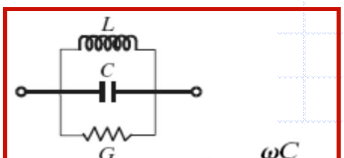


$$Q_u = \frac{\omega C}{G}$$

For a lowpass or a highpass filter, ω is usually the cutoff frequency; while for a bandpass or bandstop filter, is the center frequency.



$$Q_u = \frac{\omega L}{R}$$



$$Q_u = \frac{\omega C}{G}$$

ω is the resonant freq. for the lossy resonator

Dissipation Effect on Lowpass and Highpass Filters

Assuming that the unloaded quality factors of all reactive elements in a filter are known, determined theoretically or experimentally, we can find *R* and *G* for the lossy reactive elements from previous equations. The dissipation effects on the filter insertion loss response can easily estimated by analysis of the whole filter equivalent circuit, including the dissipative elements *R* and *G*.

Another approach based on simple formula by Cohn (1959)

$$\Delta L_{A0} = 4.343 \sum_{i=1}^{n} \frac{\Omega_c}{Q_{ui}} g_i \text{ dB}$$

where ΔL_{A0} is the dB increase in insertion loss at $\omega = 0$, Ω_c and g_i are the cutoff frequency and element values of the lowpass prototypes, as discussed previously in this chapter, and Q_{ui} are the unloaded quality factors of microwave elements corresponding to g_i, which are given at the cutoff ω_c of the practical lowpass filters. This

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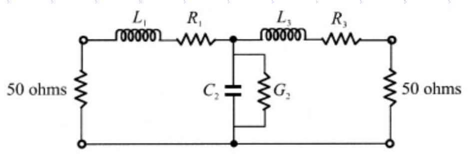
Dissipation Effect on Lowpass and Highpass Filters

As an example, let us consider the lowpass filter designed previously in Figure 3.13, which has a Butterworth response with a cutoff frequency at *fc* = 2 GHz. To take into account the finite unloaded quality factors of the reactive elements, the filter circuit becomes that of Figure below.

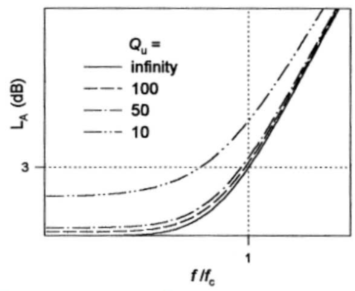
Two effects are obvious:

1. A shift of insertion loss by a constant amount determined by the additional loss at zero frequency.
2. A gradual rounding off of the insertion loss curve at the passband edge, resulting in diminished width of the passband and hence in reduced selectivity.



$L_1 = L_3 = 3.979$ nH $C_2 = 3.183$ pF

$R_1 = R_3 = \frac{2\pi f_c L_1}{Q_u}$ $G_2 = \frac{2\pi f_c C_2}{Q_u}$ $f_c = 2$ GHz



The lowpass filter has used a $n = 3$ Butterworth lowpass prototype with $g_1 = g_3 = 1$, $g_2 = 2$ and $\Omega_c = 1$. As assumed in the above example that $Q_{u1} = Q_{u2} = Q_{u3} = Q_u$, we then have $\Delta L_{A0} = 0.174$ dB for $Q_u = 100$, and $\Delta L_{A0} = 1.737$ dB for $Q_u = 10$, according to (3.61), which are in excellent agreement with the results obtained by network analysis.

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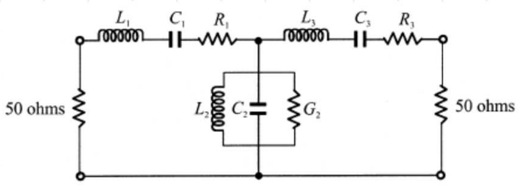
Dissipation Effect on Bandpass and Bandstop Filters

It should be mentioned that not only does the passband insertion loss increase and the selectivity become worse as the *Qu* is decreased, but it also can be shown that for a given *Qu*, the same tendencies occur as the fractional bandwidth of the filter is reduced.

$$\Delta L'_{A0} = 4.343 \sum_{i=1}^{n} \frac{\Omega_c}{FBW Q_{ui}} g_i \text{ dB}$$

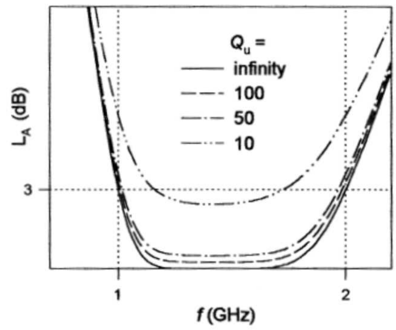
The bandpass filter has a fractional bandwidth $FBW = 0.707$ and a center frequency $f_0 = 1.414$ GHz. Again, we assume that $Q_{u1} = Q_{u2} = Q_{u3} = Q_u$. Substituting these data into (3.62) yields $\Delta L'_{A0} = 0.246$ dB for $Q_u = 100$, and $\Delta L'_{A0} = 2.457$ dB for $Q_u = 10$, which are almost the same as those obtained by network analysis, as can be seen from Figure 3.31(*b*). The expression of



$L_1 = L_3 = 7.958$ nH $\quad C_1 = C_3 = 1.592$ pF

$L_2 = 1.989$ nH $\quad C_2 = 6.366$ pF

$R_1 = R_3 = \frac{2\pi f_0 L_1}{Q_u} \quad G_2 = \frac{2\pi f_0 C_2}{Q_u} \quad f_0 = 1.414$ GHz



HW #1

- Prove equation (3.24) for Butterworth Lowpass Prototype Filter
- Prove the equations shown in Fig. 3.18(a) and Fig. 3.18(b).
- Prove the equations shown in Fig. 3.19(a) and Fig. 3.19(b).
- Please prove the reactance slope parameters of a lumped LC series resonator is $\omega_0 L$, and the susceptance slope parameters of a lumped LC parallel resonator is $\omega_0 C$.
- Prove the equations shown in Fig. 3.20(a) and Fig. 3.20(b).
- Please prove Fig. 3.21(a) & (d) have a phase shift of +90 degree, and Fig. 3.21(b) & (c) have a phase shift of -90 degree.
- Please prove the identities in Fig. 3.27.