# Several Topics for Electronics and Photonics

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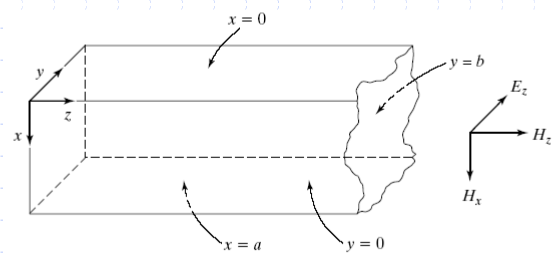
# Introduction

In the previous chapter, we introduced the principles of guided waves and learned that the mechanism of waveguiding is one in which the waves bounce obliquely between parallel planes as they progress along the structure. We studied transverse electric (TE) and transverse magnetic (TM) waves supported by plane conductors, as well as those supported by a plane dielectric slab.

Thus, we restricted our study of guided waves to one-dimensional structures.

In this chapter, we extend the treatment to two dimensions.

## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR






## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

*Derivation of field expressions for TE modes*

By making use of the expansions for the Maxwell's curl equations in Cartesian coordinates, that all transverse (*x* and *y*) field components are derivable from the longitudinal field component $H_z$

$$j\beta_z\bar{E}_y = -j\omega\mu\bar{H}_x \qquad \frac{\partial\bar{H}_z}{\partial y} + j\beta_z\bar{H}_y = j\omega\varepsilon\bar{E}_x$$

$$-j\beta_z\bar{E}_x = -j\omega\mu\bar{H}_y \qquad -j\beta_z\bar{H}_x - \frac{\partial\bar{H}_z}{\partial x} = j\omega\varepsilon\bar{E}_y$$

$$\frac{\partial\bar{E}_y}{\partial x} - \frac{\partial\bar{E}_x}{\partial y} = -j\omega\mu\bar{H}_z \qquad \frac{\partial\bar{H}_y}{\partial x} - \frac{\partial\bar{H}_x}{\partial y} = 0$$

$$\bar{E}_x = \frac{j\omega\mu}{\beta_z^2 - \beta^2}\frac{\partial\bar{H}_z}{\partial y}$$

$$\bar{E}_y = -\frac{j\omega\mu}{\beta_z^2 - \beta^2}\frac{\partial\bar{H}_z}{\partial x}$$

$$\bar{H}_x = j\frac{\beta_z}{\beta_z^2 - \beta^2}\frac{\partial\bar{H}_z}{\partial x}$$

$$\bar{H}_y = j\frac{\beta_z}{\beta_z^2 - \beta^2}\frac{\partial\bar{H}_z}{\partial y}$$

obtain a differential equation for $\bar{H}_z$ as given by

$$\boxed{\frac{\partial^2 \bar{H}_z}{\partial x^2} + \frac{\partial^2 \bar{H}_z}{\partial y^2} - (\beta_z^2 - \beta^2)\bar{H}_z = 0}$$

Recall that $\beta = \omega\sqrt{\mu\varepsilon}$, so that $\beta^2 = \omega^2\mu\varepsilon$.

we make use of the *separation of variables* technique

This consists of assuming that the required function of the two variables *x* and *y* is the product of two functions, one of which is a function of *x* only and the second is a function of *y* only.

$$\bar{H}_z(x, y, z) = \bar{X}(x)\bar{Y}(y)e^{-j\beta_z z}$$

Substituting (9.7) into (9.6), we then obtain

$$\bar{X}''(x)\bar{Y}(y) + \bar{X}(x)\bar{Y}''(y) - (\beta_z^2 - \beta^2)\bar{X}(x)\bar{Y}(y) = 0$$

or

$$\frac{\bar{X}''}{\bar{X}} + \frac{\bar{Y}''}{\bar{Y}} = \beta_z^2 - \beta^2$$



9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

Equation (9.8) says that a function of *x* only plus a function of *y* only is equal to a constant. For this to be satisfied, both functions must be equal to constants.

$$\frac{\bar{X}''}{\bar{X}} = -\beta_x^2, \text{ a constant} \qquad \frac{\bar{Y}''}{\bar{Y}} = -\beta_y^2, \text{ a constant}$$

$$\longrightarrow \frac{d^2\bar{X}}{dx^2} = -\beta_x^2\bar{X} \qquad \longrightarrow \frac{d^2\bar{Y}}{dy^2} = -\beta_y^2\bar{Y}$$

We have thus obtained two ordinary differential equations involving separately the two variables *x* and *y*; hence, the technique is known as the *separation of variables* technique.

The solutions to (9.10a) and (9.10b) are given by

$$\bar{X}(x) = \bar{A}_1 e^{j\beta_x x} + \bar{A}_2 e^{-j\beta_x x}$$

$$\bar{Y}(y) = \bar{B}_1 e^{j\beta_y y} + \bar{B}_2 e^{-j\beta_y y}$$

$$\longrightarrow \bar{H}_z = (\bar{A}_1 e^{j\beta_x x} + \bar{A}_2 e^{-j\beta_x x})(\bar{B}_1 e^{j\beta_y y} + \bar{B}_2 e^{-j\beta_y y})e^{-j\beta_z z}$$

where $\bar{A}_1$, $\bar{A}_2$, $\bar{B}_1$, and $\bar{B}_2$ are constants.

## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

We also note from substitution of (9.9a) and (9.9b) into (9.8) that

$$\beta_z^2 = \beta^2 - \beta_x^2 - \beta_y^2$$

To determine the constants in (9.11), we make use of the boundary conditions that require that the tangential components of the electric-field intensity on all four walls of the guide be zero.

$$\begin{aligned} \bar{E}_x &= 0 \quad \text{for} \quad y = 0, 0 < x < a \\ \bar{E}_x &= 0 \quad \text{for} \quad y = b, 0 < x < a \\ \bar{E}_y &= 0 \quad \text{for} \quad x = 0, 0 < y < b \\ \bar{E}_y &= 0 \quad \text{for} \quad x = a, 0 < y < b \end{aligned}$$

→

$$\begin{aligned} \frac{\partial \bar{H}_z}{\partial y} &= 0 \quad \text{for} \quad y = 0, 0 < x < a \\ \frac{\partial \bar{H}_z}{\partial y} &= 0 \quad \text{for} \quad y = b, 0 < x < a \\ \frac{\partial \bar{H}_z}{\partial x} &= 0 \quad \text{for} \quad x = 0, 0 < y < b \\ \frac{\partial \bar{H}_z}{\partial x} &= 0 \quad \text{for} \quad x = a, 0 < y < b \end{aligned}$$



## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

Using (9.13c) and (9.13a) in conjunction with (9.11), we get

$$\begin{aligned} \bar{A}_1 - \bar{A}_2 = 0 \quad &\text{or} \quad \bar{A}_2 = \bar{A}_1 \\ \bar{B}_1 - \bar{B}_2 = 0 \quad &\text{or} \quad \bar{B}_2 = \bar{B}_1 \end{aligned}$$

$$\bar{H}_z = \bar{A} \cos \beta_x x \cos \beta_y y \, e^{-j\beta_z z}$$

where $\bar{A}$ is a constant. Using the remaining two boundary conditions (9.13d) and (9.13b), we then obtain

$$\sin \beta_x a = 0 \quad \text{or} \quad \beta_x = \frac{m\pi}{a}, \quad m = 0, 1, 2, \ldots \qquad (9.15a)$$

$$\sin \beta_y b = 0 \quad \text{or} \quad \beta_y = \frac{n\pi}{b}, \quad n = 0, 1, 2, \ldots \qquad (9.15b)$$

Thus, the solution for $\bar{H}_z$ for the $TE_{m,n}$ mode is given by

→ $$\boxed{\bar{H}_z = \bar{A} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta_z z}}$$

→ $$\beta_z^2 = \beta^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

For propagation to occur, the exponent $\beta_z$ in (9.16) must be real. Hence, the cut-off condition is given by

$$\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 = 0 \qquad (9.18)$$

the cutoff frequency is given by

$$f_c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}$$

the cutoff wavelength is

$$\lambda_c = \frac{1}{\sqrt{\mu\varepsilon}f_c} = \frac{1}{\sqrt{(m/2a)^2 + (n/2b)^2}}$$



## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

$$\beta_z^2 - \beta^2 = -\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 = -(2\pi)^2\left[\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2\right] = -\left(\frac{2\pi}{\lambda_c}\right)^2$$

$$\bar{E}_x = j\frac{\omega\mu\lambda_c^2}{4\pi^2}\frac{n\pi}{b}\bar{A}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-j\beta_z z}$$

$$\bar{E}_y = -j\frac{\omega\mu\lambda_c^2}{4\pi^2}\frac{m\pi}{a}\bar{A}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-j\beta_z z}$$

$$\bar{H}_x = j\frac{\lambda_c^2}{2\pi\lambda_g}\frac{m\pi}{a}\bar{A}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-j\beta_z z}$$

$$\bar{H}_y = j\frac{\lambda_c^2}{2\pi\lambda_g}\frac{n\pi}{b}\bar{A}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-j\beta_z z}$$

It can also be seen that if both *m* and *n* are equal to zero, then all transverse field components go to zero.

Therefore, *for TE modes, either m or n can be zero, but both m and n cannot be zero*

## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

The entire procedure for the derivation of the field expressions can be repeated for TM waves by starting with the longitudinal field component Ez.

Therefore, *for TM modes both m and n must be nonzero.*



TABLE 9.1 Field Expressions and Associated Parameters for TE and TM Modes in a Rectangular Waveguide

| Transverse electric (TE) waves | Transverse magnetic (TM) waves |
|---|---|
| Field Expressions: | Field Expressions: |
| $(m, n = 0, 1, 2, \ldots$, but not both zero) | $(m, n = 1, 2, 3, \ldots)$ |
| $\bar{E}_z = 0$ | $\bar{H}_z = 0$ |
| $\bar{H}_z = \bar{A}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{\mp j\beta_z z}$ | $\bar{E}_z = \bar{A}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{\mp j\beta_z z}$ |
| $\bar{E}_x = j\frac{\lambda_c^2}{4\pi^2}\omega\mu\frac{n\pi}{b}\bar{A}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{\mp j\beta_z z}$ | $\bar{E}_x = \mp j\frac{\lambda_c^2}{2\pi\lambda_g}\frac{m\pi}{a}\bar{A}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{\mp j\beta_z z}$ |
| $\bar{E}_y = -j\frac{\lambda_c^2}{4\pi^2}\omega\mu\frac{m\pi}{a}\bar{A}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{\mp j\beta_z z}$ | $\bar{E}_y = \mp j\frac{\lambda_c^2}{2\pi\lambda_g}\frac{n\pi}{b}\bar{A}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{\mp j\beta_z z}$ |
| $\bar{H}_x = \mp\frac{\bar{E}_y}{\eta_g}$ | $\bar{H}_x = \mp\frac{\bar{E}_y}{\eta_g}$ |
| $\bar{H}_y = \pm\frac{\bar{E}_x}{\eta_g}$ | $\bar{H}_y = \pm\frac{\bar{E}_x}{\eta_g}$ |
| $f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ | $f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ |
| $\lambda_c = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}}$ | $\lambda_c = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}}$ |
| $\lambda_g = \frac{\lambda}{\sqrt{1 - (\lambda/\lambda_c)^2}} = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}}$ | $\lambda_g = \frac{\lambda}{\sqrt{1 - (\lambda/\lambda_c)^2}} = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}}$ |
| $v_{pz} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1 - (f_c/f)^2}} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1 - (\lambda/\lambda_c)^2}}$ | $v_{pz} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1 - (f_c/f)^2}} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1 - (\lambda/\lambda_c)^2}}$ |
| $\eta_g = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - (f_c/f)^2}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - (\lambda/\lambda_c)^2}}$ | $\eta_g = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}$ |

## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

*Dominant mode*

Waveguides are, however, designed so that only one mode, the mode with the lowest cutoff frequency (or the largest cutoff wavelength), propagates. This is known as the *dominant mode.*

From Table 9.1, we can see that the dominant mode is the TE1,0 mode or the TE0,1 mode, depending on whether the dimension *a* or the dimension *b* is the larger of the two.

By convention, the larger dimension is designated to be *a*, and hence the mode is the dominant mode.



**Example 9.2 Application of transmission-line analogy to a rectangular waveguide system**

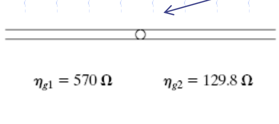
A rectangular waveguide extending in the $z$-direction and having the dimensions $a = 4$ cm and $b = 2$ cm has a dielectric discontinuity at $z = 0$, as shown in Fig. 9.4. For $TE_{1,0}$ waves of frequency $f = 5000$ MHz incident from section 1, we wish to find (a) the transmission-line equivalent and (b) the length and the permittivity of a quarter-wave section required to achieve a match between the two sections.

**(a)** First, we note that for the $TE_{1,0}$ mode, $\lambda_c = 2a = 8$ cm for both sections. For $f = 5000$ MHz, the wavelength in free space is $\lambda_1 = 6$ cm and the wavelength in a dielectric of permittivity $9\varepsilon_0$ is $\lambda_2 = 2$ cm. Since $\lambda_1$ and $\lambda_2$ are both less than $\lambda_c$, the $TE_{1,0}$ mode propagates in both sections. Denoting the guide parameters associated with sections 1 and 2 by subscripts 1 and 2, respectively, we then obtain

$$\eta_{g1} = \frac{\eta_1}{\sqrt{1-(\lambda_1/\lambda_c)^2}} = \frac{377}{\sqrt{1-(6/8)^2}} = 570\ \Omega$$

$$\eta_{g2} = \frac{\eta_2}{\sqrt{1-(\lambda_2/\lambda_c)^2}} = \frac{377/3}{\sqrt{1-(2/8)^2}} = 129.8\ \Omega$$

Thus, the transmission-line equivalent is as shown in Fig. 9.5.



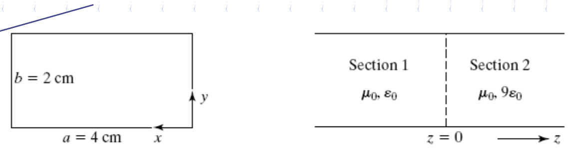




FIGURE 9.4
Rectangular waveguide discontinuity.

**(b)** The characteristic impedance of a quarter-wave section required to achieve a match between line 1 and line 2 must be equal to $\sqrt{\eta_{g1}\eta_{g2}}$. Denoting the parameters associated with the quarter-wave section by subscript 3, we then have

$$\eta_{g3} = \frac{\eta_3}{\sqrt{1 - (\lambda_3/\lambda_c)^2}} = \sqrt{\eta_{g1}\eta_{g2}}$$

or

$$\frac{\eta_1\sqrt{\varepsilon_0/\varepsilon_3}}{\sqrt{1 - (\lambda_1/\lambda_c)^2(\varepsilon_0/\varepsilon_3)}} = \sqrt{\eta_{g1}\eta_{g2}}$$

$$\frac{\varepsilon_0/\varepsilon_3}{1 - (6/8)^2(\varepsilon_0/\varepsilon_3)} = \frac{570 \times 129.8}{(377)^2} = 0.5205$$

solving which we obtain $\varepsilon_3 = 2.484\varepsilon_0$. To find the length of the quarter-wave section, we compute

$$\lambda_{g3} = \frac{\lambda_3}{\sqrt{1 - (\lambda_3/\lambda_c)^2}} = \frac{\lambda_1\sqrt{\varepsilon_0/\varepsilon_3}}{\sqrt{1 - (\lambda_1/\lambda_c)^2(\varepsilon_0/\varepsilon_3)}}$$

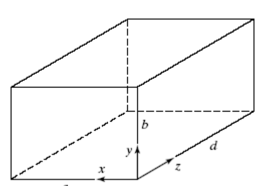
$$= \frac{6 \times 0.6345}{\sqrt{1 - (9/16) \times 0.4026}} = 4.33 \text{ cm}$$

Thus, the length of the quarter-wave section is $\lambda_{g3}/4$, or 1.0825 cm.



## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

### cavity resonator




Let us now consider guided waves of equal amplitude propagating in the positive $z$- and negative $z$-directions in a rectangular waveguide.

This can be achieved by terminating the guide by a perfectly conducting sheet in a constant $z$ plane, that is, a transverse plane of the guide.

Due to perfect reflection from the sheet, the fields will then be characterized by standing wave nature along the guide axis, that is, in the $z$-direction, in addition to the standing wave nature in the $x$- and $y$-directions.

The standing wave pattern along the guide axis will have nulls of transverse electric field on the terminating sheet and in planes parallel to it at distances of integer multiples of $\lambda_g/2$.

## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

**cavity resonator**

Such a structure is known as a *cavity resonator* and is the counterpart of the low-frequency lumped parameter resonant circuit at microwave frequencies, since it supports oscillations at frequencies for which the foregoing condition, that is,

$$d = l\frac{\lambda_g}{2}, \qquad l = 1, 2, 3, \ldots$$

. Substituting for $\lambda_g$ in (9.23) from Table 9.1 and rearranging, we

$$\frac{2d}{l} = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}} \longrightarrow \frac{1}{\lambda^2} - \frac{1}{\lambda_c^2} = \left(\frac{l}{2d}\right)^2$$

which upon substitution for $\lambda_c$ gives

$$\frac{1}{\lambda^2} = \left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 + \left(\frac{l}{2d}\right)^2$$

$$\lambda = \frac{1}{\sqrt{(m/2a)^2 + (n/2b)^2 + (l/2d)^2}}$$



## 9.1 RECTANGULAR METALLIC WAVEGUIDE AND CAVITY RESONATOR

**cavity resonator**

The expression for the frequencies of oscillation is thus given by

$$f_{osc} = \frac{v_p}{\lambda} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 + \left(\frac{l}{2d}\right)^2}$$

## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

*Loss in dielectric*

Power dissipation in the imperfect dielectric of a guide results in loss that follows simply from the attenuation constant for the case of a uniform plane wave propagating in the dielectric.

We consider the TE or TM wave in a parallel-plate waveguide, then we know that progress of the composite TE or TM wave along the guide by a distance $d$ involves travel of the component uniform plane waves obliquely to the plates by a distance $d/\sqrt{1-(f_c/f)^2}$.

Thus, if $\alpha_d'$ is the attenuation constant for uniform plane wave propagation in the dielectric, then the attenuation constant $\alpha_d$ for the TE or TM wave along the guide axis is $\alpha_d'/\sqrt{1-(f_c/f)^2}$ and the attenuation $e^{\mp\alpha_d z}$ is equal to $e^{\mp[\alpha_d'/\sqrt{1-(f_c/f)^2}]z}$. From

Section 4.5, we recall that for a slightly imperfect dielectric ($\sigma/\omega\varepsilon \ll 1$),

$$\alpha_d' \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\left(1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right) \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$



## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

*Basis for analysis of loss in conductors*

The procedure is based on considering the situation as though a plane wave having the same magnetic field components as those given by the appropriate tangential magnetic field components on that wall for the perfect conductor case propagates normally into the conductor and

then computing the power flow into the wall (assumed to be of infinite depth in view of the rapid attenuation of fields as they propagate into a good conductor).

Now, for a tangential magnetic field $\overline{\mathbf{H}}_t$ on a given wall, the electric-field vector of a uniform plane wave propagating into the wall (designated to be in the direction $\mathbf{a}_n$) is $\overline{\eta}_c\overline{\mathbf{H}}_t \times \mathbf{a}_n$, where $\overline{\eta}_c$ is the intrinsic impedance of the conductor. The complex Poynting vector is

$$\begin{aligned}\overline{\mathbf{P}} &= \tfrac{1}{2}\overline{\mathbf{E}} \times \overline{\mathbf{H}}^* = \tfrac{1}{2}\overline{\eta}_c(\overline{\mathbf{H}}_t \times \mathbf{a}_n) \times \overline{\mathbf{H}}_t^* \\ &= \tfrac{1}{2}\overline{\eta}_c[\mathbf{a}_n(\overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^*) - \overline{\mathbf{H}}_t(\mathbf{a}_n \cdot \overline{\mathbf{H}}_t^*)] \\ &= \tfrac{1}{2}\overline{\eta}_c\overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^*\,\mathbf{a}_n\end{aligned}$$

## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

*Basis for analysis of loss in conductors*

The time-average power flowing into the conductor of conductivity $\sigma$ for a length $\Delta z$ along the guide is given by

$$\Delta\langle P_d\rangle = \int_l (\mathrm{Re}\,\overline{\mathbf{P}})\cdot dl\,\Delta z\,\mathbf{a}_n$$
$$= \int_l \mathrm{Re}\left(\frac{1}{2}\overline{\eta}_c\,\overline{\mathbf{H}}_t\cdot\overline{\mathbf{H}}_t^*\right) dl\,\Delta z$$
$$= \frac{\Delta z}{2\sigma\delta}\int_l \overline{\mathbf{H}}_t\cdot\overline{\mathbf{H}}_t^*\,dl$$

where $\delta(=1/\sqrt{\pi f\mu\sigma})$ is the skin depth at the frequency of operation $f$, $dl$ is the differential length element along the transverse dimension, and $\int_l$ denotes integration performed along the transverse dimension. We shall illustrate the ap-



## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: Attenuation constant for mode in a rectangular guide with imperfect conductors

To obtain the attenuation constant $\alpha_c$, we note that since for a given mode the fields are attenuated in the manner $e^{-\alpha_c z}$ where the $z$-direction is assumed to be the guide axis, the time-average power flow $\langle P_f\rangle$ down the guide varies in the manner $e^{-2\alpha_c z}$. The

The time-average power dissipated over an infinitesimal distance Δz at any value of *z* along the guide is then given by

$$\Delta\langle P_d\rangle = -\frac{\partial\langle P_f\rangle}{\partial z}\Delta z$$
$$= 2\alpha_c\langle P_f\rangle\,\Delta z$$

$$\boxed{\alpha_c = \frac{1}{2\langle P_f\rangle}\frac{\Delta\langle P_d\rangle}{\Delta z}}$$

## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: Attenuation constant for mode in a rectangular guide with imperfect conductors

To find $\Delta\langle \dot{P}_d \rangle$

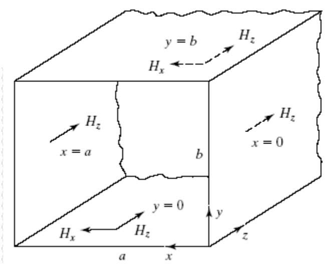
$TE_{1,0}$ mode field components in a lossless waveguide to be

$$\bar{E}_z = \bar{E}_x = \bar{H}_y = 0$$

$$\bar{H}_z = \bar{A} \cos \frac{\pi x}{a} e^{-j\beta_z z}$$

$$\bar{E}_y = -j \frac{\lambda_c^2}{4\pi^2} \omega\mu \frac{\pi}{a} \bar{A} \sin \frac{\pi x}{a} e^{-j\beta_z z}$$

$$\bar{H}_x = -\frac{\bar{E}_y}{\eta_g} = j\frac{2a}{\lambda_g} \bar{A} \sin \frac{\pi x}{a} e^{-j\beta_z z}$$






## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: Attenuation constant for mode in a rectangular guide with imperfect conductors

Each nonzero tangential component of magnetic field on a given wall will be accompanied by a tangential electric field perpendicular to it so as to produce power flow into the conductor.

Since some of these tangential electric-field components are longitudinal, the mode is no longer exactly TE mode.

However, these components are very small in magnitude; hence, the mode is almost a TE mode.

## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: Attenuation constant for mode in a rectangular guide with imperfect conductors

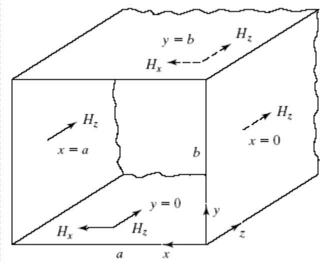
**RIGHT SIDE WALL ($x = 0$)**

$$\overline{\mathbf{H}}_t = \bar{A}\,\mathbf{a}_z$$
$$\overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^* = |\bar{A}|^2$$
$$\Delta\langle P_d\rangle = \frac{\Delta z}{2\sigma\delta}\int_{y=0}^{b} |\bar{A}|^2\,dy$$
$$= \frac{|\bar{A}|^2 b}{2\sigma\delta}\Delta z$$

**LEFT SIDE WALL ($x = a$)**

Same as for the right side wall.

$$\Delta\langle P_d\rangle = \frac{|\bar{A}|^2 b 2\sigma\delta}{\Delta z}$$






## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: Attenuation constant for mode in a rectangular guide with imperfect conductors

**BOTTOM WALL ($y = 0$)**

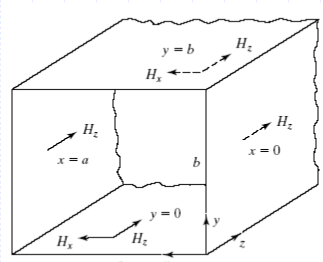
$$\overline{\mathbf{H}}_t = \bar{A}\left(j\frac{2a}{\lambda_g}\sin\frac{\pi x}{a}\mathbf{a}_x + \cos\frac{\pi x}{a}\mathbf{a}_z\right)$$
$$\overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^* = |\bar{A}|^2\left(\frac{4a^2}{\lambda_g^2}\sin^2\frac{\pi x}{a} + \cos^2\frac{\pi x}{a}\right)$$

$$\Delta\langle P_d\rangle = \frac{\Delta z}{2\sigma\delta}\int_{x=0}^{a}|\bar{A}|^2\left(\frac{4a^2}{\lambda_g^2}\sin^2\frac{\pi x}{a} + \cos^2\frac{\pi x}{a}\right)dx$$
$$= \frac{|\bar{A}|^2\Delta z}{4\sigma\delta}\left(\frac{4a^3}{\lambda_g^2} + a\right)$$

**TOP WALL ($y = b$)**

Same as for the bottom wall.

$$\Delta\langle P_d\rangle = \frac{|\bar{A}|^2\Delta z}{4\sigma\delta}\left(\frac{4a^3}{\lambda_g^2} + a\right)$$



ex: Attenuation constant for mode in a rectangular guide with imperfect conductors

Total dissipated power

$$\Delta\langle P_d\rangle = \frac{|\bar{A}|^2}{2\sigma\delta}\left(\frac{4a^3}{\lambda_g^2} + a + 2b\right)\Delta z$$
$$= \frac{|\bar{A}|^2}{2\sigma\delta}\left[\frac{4a^3}{\lambda^2}\left(1 - \frac{\lambda^2}{4a^2}\right) + a + 2b\right]\Delta z$$
$$= \frac{|\bar{A}|^2}{\sigma\delta}\left(\frac{2a^3}{\lambda^2} + b\right)\Delta z$$





ex: Attenuation constant for mode in a rectangular guide with imperfect conductors

Now, to find the time-average power transmitted down the guide, we note that the time-average Poynting vector is given by

$$\begin{aligned}\langle \mathbf{P}\rangle &= \tfrac{1}{2}\,\mathrm{Re}\,\overline{\mathbf{E}} \times \overline{\mathbf{H}}^* \\ &= -\frac{1}{2\eta_g}\bar{E}_y\bar{E}_y^*\,\mathbf{a}_z \\ &= \frac{1}{2\eta_g}\frac{\lambda_c^4}{16\pi^4}\omega^2\mu^2\frac{\pi^2}{a^2}|\bar{A}|^2\sin^2\frac{\pi x}{a}\mathbf{a}_z \\ &= \frac{\omega^2\mu^2 a^2}{2\eta_g\pi^2}|\bar{A}|^2\sin^2\frac{\pi x}{a}\mathbf{a}_z\end{aligned} \qquad (9.57)$$

The time-average power transmitted down the guide is then given by

$$\begin{aligned}\langle P_f\rangle &= \int_{y=0}^{b}\int_{x=0}^{a}\langle \mathbf{P}\rangle \cdot dx\,dy\,\mathbf{a}_z \\ &= \frac{\omega^2\mu^2 a^2}{2\eta_g\pi^2}|\bar{A}|^2\int_{y=0}^{b}\int_{x=0}^{a}\sin^2\frac{\pi x}{a}\,dx\,dy \\ &= \frac{\omega^2\mu^2 a^2 b}{4\eta_g\pi^2}|\bar{A}|^2\end{aligned}$$

## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: Attenuation constant for mode in a rectangular guide with imperfect conductors

Finally, the attenuation constant is given by

$$\begin{aligned}\alpha_c &= \frac{\Delta\langle P_d\rangle}{2\langle P_f\rangle\Delta z} \\ &= \frac{|\bar{A}|^2}{2\sigma\delta}\left(\frac{2a^3}{\lambda^2}+b\right)\frac{4\eta_g\pi^2}{\omega^2\mu^2a^3b|\bar{A}|^2} \\ &= \frac{1}{\sigma\delta}\left(1+\frac{b\lambda^2}{2a^3}\right)\frac{\eta\varepsilon}{\mu b\sqrt{1-(f_c/f)^2}} \\ &= \frac{1}{\sigma\delta}\left[1+\frac{2b}{a}\left(\frac{\lambda}{\lambda_c}\right)^2\right]\frac{1}{b\eta\sqrt{1-(f_c/f)^2}}\end{aligned}$$

$$\boxed{\alpha_c = \frac{1}{\alpha\delta b\eta\sqrt{1-(f_c/f)^2}}\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right]}$$



## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: Attenuation constant for mode in a rectangular guide with imperfect conductors

Note that for $f \to f_c, \alpha_c \to \infty$. For $f \ll f_c$, $\alpha_c \approx 1/\sigma\delta b\eta \propto \sqrt{f}$, so that for $f \to \infty$, $\alpha_c \to \infty$. Thus, as $f$ varies from $f_c$ to infinity, $\alpha_c$ varies from infinity to some minimum value and then increases to infinity. The minimum value of $\alpha_c$ occurs for

$$\left(\frac{f}{f_c}\right)^2 = \left(1.5+\frac{3b}{a}\right)+\sqrt{\left(1.5+\frac{3b}{a}\right)^2-\frac{2b}{a}} \qquad (9.60)$$

For example, for $b/a = 1/2$, the minimum value occurs for $f/f_c \approx 2.4142$.

## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

*Q factor of a resonator*

The *Q* factor, which is a measure of the frequency selectivity of the resonator, is defined as

$$Q = 2\pi \frac{\text{energy stored in the resonator}}{\text{energy dissipated per cycle}} = 2\pi f \frac{\text{energy stored in the resonator}}{\text{time average power dissipated}}$$

The power dissipated in them can be computed by analysis, as in Example 9.6 for the waveguide case.

As for the energy stored in the cavity, it is distributed between the electric and magnetic fields at any arbitrary instant of time.

But there are particular values of time at which the electric field is maximum and the magnetic field is zero, and vice versa. At these values of time, the entire energy is stored in one of the two fields.



## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: *Q* factor for $TE_{1,0,1}$ mode in a rectangular cavity resonator

First, we obtain the expressions for $TE_{1,0,1}$ mode field components by superimposing the (+) and (−) wave field components for the $TE_{1,0}$ waves from Table 9.1 and satisfying the boundary conditions of zero tangential electric fields at the ends $z = 0$ and $z = d$. Thus, we have

$$\bar{E}_z = \bar{E}_x = \bar{H}_y = 0$$

$$\bar{H}_z = \cos\frac{\pi x}{a}[\bar{A}_1 e^{-j\beta_z z} + \bar{A}_2 e^{j\beta_z z}]$$

$$\bar{E}_y = -j\frac{\lambda_c^2}{4\pi^2}\omega\mu\frac{\pi}{a}\sin\frac{\pi x}{a}[\bar{A}_1 e^{-j\beta_z z} + \bar{A}_2 e^{j\beta_z z}]$$

$$\bar{H}_x = j\frac{1}{\eta_g}\frac{\lambda_c^2}{4\pi^2}\omega\mu\frac{\pi}{a}\sin\frac{\pi x}{a}[\bar{A}_1 e^{-j\beta_z z} - \bar{A}_2 e^{j\beta_z z}]$$

$$[\bar{E}_y]_{z=0} = [\bar{E}_y]_{z=d} = 0$$

$$\bar{A}_1 + \bar{A}_2 = 0 \quad \text{or} \quad \bar{A}_2 = -\bar{A}_1$$

$$\sin\beta_z d = 0 \quad \text{or} \quad \beta_z = \pi/d$$

## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: $Q$ factor for $TE_{1,0,1}$ mode in a rectangular cavity resonator

$$\bar{E}_z = \bar{E}_x = \bar{H}_y = 0$$

$$\bar{H}_z = \bar{A}\cos\frac{\pi x}{a}\sin\frac{\pi z}{d}$$

$$\bar{E}_y = -j\bar{A}\frac{\omega\mu a}{\pi}\sin\frac{\pi x}{a}\sin\frac{\pi z}{d}$$

$$\bar{H}_x = -\bar{A}\frac{a}{d}\sin\frac{\pi x}{a}\cos\frac{\pi z}{d}$$

where $\bar{A} = -2j\bar{A}_1$ and we have also substituted $\lambda_c = 2a$ and $\omega\mu/\eta_g = 2\pi/\lambda_g = \pi/d$.



## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: $Q$ factor for $TE_{1,0,1}$ mode in a rectangular cavity resonator

Noting that the amplitude of the only electric field component Ey which is the value of Ey at the instant of time the magnetic field throughout the cavity is zero, is given by

$$|\bar{E}_y| = |\bar{A}|\frac{\omega\mu a}{\pi}\sin\frac{\pi x}{a}\sin\frac{\pi z}{d}$$

Integrating the energy density throughout the volume of the cavity,

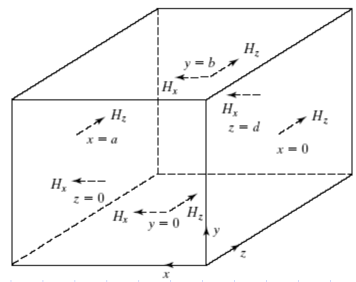
$$\begin{aligned} W_{\text{stored}} &= \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{d}\frac{1}{2}\varepsilon|\bar{E}_y|^2\,dv \\ &= \frac{1}{2}\varepsilon|\bar{A}|^2\frac{\omega^2\mu^2a^2}{\pi^2}\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{d}\sin^2\frac{\pi x}{a}\sin^2\frac{\pi z}{d}\,dx\,dy\,dz \\ &= |\bar{A}|^2\frac{\omega^2\mu^2\varepsilon}{8\pi^2}a^3bd \end{aligned}$$

## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: $Q$ factor for $TE_{1,0,1}$ mode in a rectangular cavity resonator

To find the time-average power dissipated in the walls of the cavity, we note from the application of (9.51) that for a given wall, the time-average power dissipated is

$$\begin{aligned}\langle P_d \rangle &= \int_S (\mathrm{Re}\,\overline{\mathbf{P}}) \cdot dS\, \mathbf{a}_n \\ &= \int_S \mathrm{Re}\left(\frac{1}{2}\overline{\eta}_c\, \overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^*\right) dS \\ &= \frac{1}{2\sigma\delta}\int_S \overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^*\, dS\end{aligned}$$






## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: $Q$ factor for $TE_{1,0,1}$ mode in a rectangular cavity resonator

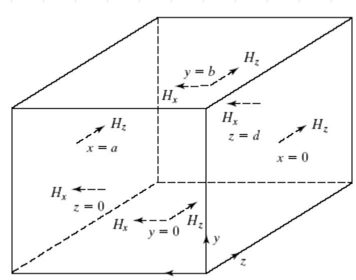
**Right Side Wall ($x = 0$)**

$$\begin{aligned}\overline{\mathbf{H}}_t &= \overline{A}\sin\frac{\pi z}{d}\mathbf{a}_z \\ \overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^* &= |\overline{A}|^2 \sin^2\frac{\pi z}{d} \\ \langle P_d \rangle &= \frac{1}{2\sigma\delta}\int_{y=0}^{b}\int_{z=0}^{d} \overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^*\, dy\, dz \\ &= \frac{|\overline{A}|^2 bd}{4\sigma\delta}\end{aligned}$$

**Left Side Wall ($x = a$)**

Same as for right side wall.

$$\langle P_d \rangle = \frac{|\overline{A}|^2 bd}{4\sigma\delta}$$



## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: $Q$ factor for $TE_{1,0,1}$ mode in a rectangular cavity resonator

**Bottom Wall ($y = 0$)**

$$\overline{\mathbf{H}}_t = \overline{A}\left[-\frac{a}{d}\sin\frac{\pi x}{a}\cos\frac{\pi z}{d}\mathbf{a}_x + \cos\frac{\pi x}{a}\sin\frac{\pi z}{d}\mathbf{a}_z\right]$$

$$\overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^* = |\overline{A}|^2\left[\left(\frac{a}{d}\right)^2 \sin^2\frac{\pi x}{a}\cos^2\frac{\pi z}{d} + \cos^2\frac{\pi x}{a}\sin^2\frac{\pi z}{d}\right]$$

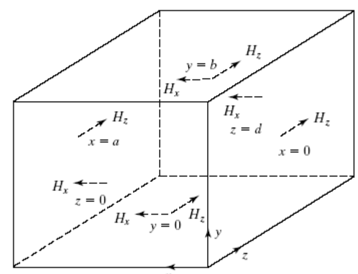
$$\langle P_d \rangle = \frac{1}{2\sigma\delta}\int_{x=0}^{a}\int_{z=0}^{d} \overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^* \, dx\, dz$$

$$= \frac{|\overline{A}|^2}{8\sigma\delta}\left(\frac{a^3}{d} + ad\right)$$

**Top Wall ($y = b$)**

Same as for bottom wall.

$$\langle P_d \rangle = \frac{|\overline{A}|^2}{8\sigma\delta}\left(\frac{a^3}{d} + ad\right)$$






## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: $Q$ factor for $TE_{1,0,1}$ mode in a rectangular cavity resonator

**Front Wall ($z = 0$)**

$$\overline{\mathbf{H}}_t = -\overline{A}\frac{a}{d}\sin\frac{\pi x}{a}\mathbf{a}_x$$

$$\overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^* = |\overline{A}|^2\left(\frac{a}{d}\right)^2 \sin^2\frac{\pi x}{a}$$

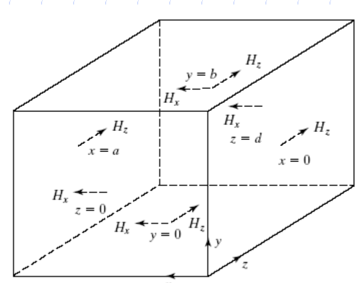
$$\langle P_d \rangle = \frac{1}{2\sigma\delta}\int_{x=0}^{a}\int_{y=0}^{b} \overline{\mathbf{H}}_t \cdot \overline{\mathbf{H}}_t^* \, dx\, dy$$

$$= \frac{|\overline{A}|^2}{4\sigma\delta}\frac{a^3 b}{d^2}$$

**Back Wall ($z = d$)**

Same as for front wall.

$$\langle P_d \rangle = \frac{|\overline{A}|^2}{4\sigma\delta}\frac{a^3 b}{d^2}$$



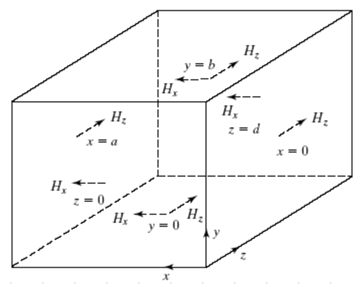
## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: $Q$ factor for $TE_{1,0,1}$ mode in a rectangular cavity resonator

Adding up the contributions from all six walls, we obtain the total time-average power dissipated to be

$$\langle P_d \rangle = \frac{|\bar{A}|^2}{2\sigma\delta}\left(bd + \frac{a^3}{2d} + \frac{ad}{2} + \frac{a^3 b}{d^2}\right) \qquad (9.70)$$

$$Q = \omega \frac{\omega^2 \mu\varepsilon a^3 bd/8\pi^2}{\frac{1}{2\sigma\delta}\left(bd + \frac{a^3}{2d} + \frac{ad}{2} + \frac{a^3 b}{d^2}\right)} = \frac{\omega^3\mu^2\varepsilon\sigma\delta}{4\pi^2}\,\frac{2a^3bd^3}{2bd^3 + a^3d + ad^3 + 2a^3b}$$






## 9.3 LOSSES IN METALLIC WAVEGUIDES AND RESONATORS

ex: $Q$ factor for $TE_{1,0,1}$ mode in a rectangular cavity resonator

From (9.25), the resonant frequency $\omega$ for the $TE_{1,0,1}$ mode is, however, given by

$$\omega = \frac{\pi}{\sqrt{\mu\varepsilon}}\left[\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2\right]^{1/2} \qquad (9.72)$$

Thus, (9.71) reduces to

$$\boxed{Q = \frac{\pi\sigma\delta\eta}{2}\,\frac{b(a^2 + d^2)^{3/2}}{ad(a^2 + d^2) + 2b(a^3 + d^3)}} \qquad (9.73)$$